A Hybrid High-Order method for Kirchhoff–Love plate bending problems

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Motivations

- Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected polygonal 2D domain

Kirchhoff–Love clamped plate problem

For $f \in L^2(\Omega)$, find $u \in H_0^2(\Omega)$ such that

$$(\mathbb{A}\boldsymbol{\nabla}^2 u, \boldsymbol{\nabla}^2 v) = (f, v) \quad \forall v \in H_0^2(\Omega)$$

- Standard conforming discretizations are computationally expensive (C^1 elements)
- HCT energy-error estimate [Ciarlet, 1974]: provided $u \in H^4(\Omega)$,

$$\|u - u_h\|_{H^2(\Omega)} \leqslant Ch^2 |u|_{H^4(\Omega)}$$

$$\Downarrow$$

- **Goal**: devising a nonconforming numerical scheme, so as to improve the computational cost of the method

♣ [B., Di Pietro, Geymonat, Krasucki, M2AN 2018]

Numerical methods for KL plates – state of the art

- [**Johnson, 1973**]: convergence of mixed FEM
- [**Lascaux–Lesaint, 1975**]: convergence of FEM on nonconforming elements
- [**Baker, 1977**]: first discontinuous element approach
- [**Brezzi–Fortin, 1991**]: mixed FEM, first-order equations
- [**Amara–Capatina–Chatti, 2002**]: mixed FEM, Tartar's lemma
- [**Mozolevski–Süli, 2003**]: nonsymmetric hp-Discontinuous Galerkin
- [**Georgoulis–Houston, 2008**]: hp-Discontinuous Galerkin, unified approach
- [**Cockburn–Dong–Guzmán, 2009**]: Hybridizable Discontinuous Galerkin
- [**Brenner, 2010**]: C^0 interior penalty
- [**Behrens–Guzmán, 2011**]: mixed FEM, first-order equations
- [**Brezzi–Marini, 2013**]: Virtual Element Methods
- [**Mu–Wang–Ye, 2014**]: Weak Galerkin
- [**Chinosi–Marini, 2016**]: VEM, L^2-estimates
- [**Zhao–Chen–Zang, 2016**]: nonconforming VEM
- [**Antonietti–Manzini–Verani, 2017**]: fully nonconforming VEM, nodal unknowns
- [**B.–DiPietro–Geymonat–Krasucki, 2018**]: Hybrid High-Order, primal
- [**Dong, 2018**]: hp-Discontinuous Galerkin method on polytopal grids

Key ideas for HHO

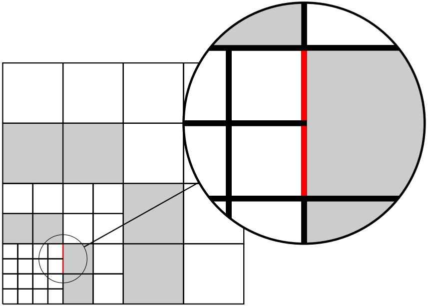
- Discrete unknowns
  - Polynomials of order $k \geqslant 1$ on mesh cells and faces
  - Cell-based unknowns can be eliminated by static condensation
- Building principles
  - Reconstruction operator based on local primal Neumann problems
  - Face-based penalty linking cell- and face-based unknowns
- Main benefits
  - Capability of handling general polygonal meshes
  - High-order method: energy-error estimate of order $(k+1)$, L^2-error estimate of order $(k+3)$ for smooth solutions
  - Key continuous mechanical properties reproduced at the discrete level

Mesh

Mesh regularity

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$, and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the usual sense of Ciarlet
- contact-regular, i.e., every simplex $S \subset T$ is s.t. $h_S \approx h_T$



Consequences [Di Pietro & Ern, 2012; Di Pietro & Droniou, 2017]:

- L^2-trace and inverse inequalities
- Approximation for broken polynomial spaces

Hypothesis: the material tensor field $\mathbb{A}$ is **element-wise constant**; we set

$$\mathbb{A}_T := \mathbb{A}_{|T} \quad \forall T \in \mathcal{T}_h$$

Projectors on local polynomial spaces

Let $X \in \mathcal{T}_h \cup \mathcal{F}_h$ be a mesh cell or face.

- The L^2-orthogonal projector $\pi_X^\ell : L^2(X) \to \mathbb{P}^\ell(X)$ is s.t.

$$(\pi_X^\ell v - v, w)_X = 0 \quad \forall w \in \mathbb{P}^\ell(X)$$

- The local energy projector $\varpi_T^\ell : H^2(T) \to \mathbb{P}^\ell(T)$, for $\ell \geqslant 2$, is s.t.

$$\left(\mathbb{A}_T \boldsymbol{\nabla}^2(\varpi_T^\ell v - v), \boldsymbol{\nabla}^2 w\right)_T = 0 \quad \forall w \in \mathbb{P}^\ell(T),$$
$$\pi_T^1(\varpi_T^\ell v - v) = 0$$

- Both projectors have **optimal approximation properties** in $H^s(T)$

Theorem (Optimal approximation properties of ϖ_T^ℓ)

There is $C > 0$ independent of h, but possibly depending on $\mathbb{A}$, s.t., for all $T \in \mathcal{T}_h$, all $s \in \{2, \ldots, \ell+1\}$, and all $v \in H^s(T)$,

$$|v - \varpi_T^\ell v|_{H^m(T)} \leqslant C h_T^{s-m} |v|_{H^s(T)} \qquad \forall m \in \{0, \ldots, s-1\},$$
$$|v - \varpi_T^\ell v|_{H^m(\partial T)} \leqslant C h_T^{s-m-1/2} |v|_{H^s(T)} \qquad \forall m \in \{0, \ldots, s-1\}$$

Local unknowns and interpolation

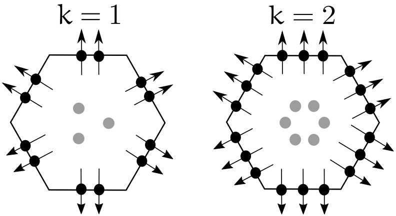


Figure: $\underline{\mathrm{U}}_T^k$ for $k \in \{1,2\}$

- For $k \geqslant 1$, and $T \in \mathcal{T}_h$, we define the local space of discrete unknowns

$$\boxed{\begin{gathered}\underline{\mathrm{U}}_T^k \coloneqq \mathbb{P}^k(T) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F)^2 \right) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right) \\ \underline{\mathrm{u}}_T \equiv (u_T, (\boldsymbol{v}_{\boldsymbol{\nabla},F})_{F \in \mathcal{F}_T}, (v_F)_{F \in \mathcal{F}_T})\end{gathered}}$$

- The local interpolator $\underline{\mathrm{I}}_T^k : H^2(T) \to \underline{\mathrm{U}}_T^k$ is s.t.

$$\underline{\mathrm{I}}_T^k v \coloneqq \left(\pi_T^k v, \left(\boldsymbol{\pi}_F^k \left((\boldsymbol{\nabla} v)_{|F}\right)\right)_{F \in \mathcal{F}_T}, (\pi_F^k (v_{|F}))_{F \in \mathcal{F}_T}\right)$$

Local deflection reconstruction I

- The local deflection reconstruction $r_T^{k+2} : \underline{U}_T^k \to \mathbb{P}^{k+2}(T)$ is s.t.

$$
\begin{aligned}
(\mathbb{A}_T \boldsymbol{\nabla}^2 r_T^{k+2} \underline{v}_T, \boldsymbol{\nabla}^2 w)_T = (v_T, \operatorname{div} \mathbf{div}\, \mathbb{A}_T \boldsymbol{\nabla}^2 w)_T \\
+ \sum_{F \in \mathcal{F}_T} \left(\boldsymbol{v}_{\boldsymbol{\nabla},F}, (\mathbb{A}_T \boldsymbol{\nabla}^2 w) \boldsymbol{n}_{TF} \right)_F \\
- \sum_{F \in \mathcal{F}_T} \left(v_F, \mathbf{div}\, \mathbb{A}_T \boldsymbol{\nabla}^2 w \cdot \boldsymbol{n}_{TF} \right)_F
\end{aligned}
$$

for all $w \in \mathbb{P}^{k+2}(T)$, with **closure condition**

$$
\pi_T^1 (r_T^{k+2} \underline{v}_T - v_T) = 0
$$

Local deflection reconstruction I

- The local deflection reconstruction $r_T^{k+2} : \underline{\mathrm{U}}_T^k \to \mathbb{P}^{k+2}(T)$ is s.t.

$$
\boxed{\begin{aligned}
(\mathbb{A}_T \boldsymbol{\nabla}^2 r_T^{k+2} \underline{\mathrm{v}}_T, \boldsymbol{\nabla}^2 w)_T &= (v_T, \operatorname{div} \mathbf{div}\, \mathbb{A}_T \boldsymbol{\nabla}^2 w)_T \\
&\quad + \sum_{F \in \mathcal{F}_T} \left(\boldsymbol{v}_{\boldsymbol{\nabla},F}, (\mathbb{A}_T \boldsymbol{\nabla}^2 w) \boldsymbol{n}_{TF} \right)_F \\
&\quad - \sum_{F \in \mathcal{F}_T} \left(v_F, \mathbf{div}\, \mathbb{A}_T \boldsymbol{\nabla}^2 w \cdot \boldsymbol{n}_{TF} \right)_F \\
&\text{for all } w \in \mathbb{P}^{k+2}(T), \text{ with } \textbf{closure condition} \\
\pi_T^1 (r_T^{k+2} \underline{\mathrm{v}}_T - v_T) &= 0
\end{aligned}}
$$

- By the definition of $\underline{\mathrm{I}}_T^k$ it holds, for all $v \in H^2(T)$ and all $w \in \mathbb{P}^{k+2}(T)$,

$$
\begin{aligned}
(\mathbb{A}_T \boldsymbol{\nabla}^2 r_T^{k+2} \underline{\mathrm{I}}_T^k v, \boldsymbol{\nabla}^2 w)_T &= (\cancel{\pi_T^k} v, \operatorname{div} \mathbf{div}\, \mathbb{A}_T \boldsymbol{\nabla}^2 w)_T \\
&\quad + \sum_{F \in \mathcal{F}_T} \left(\cancel{\boldsymbol{\pi}_F^k} (\boldsymbol{\nabla} v), (\mathbb{A}_T \boldsymbol{\nabla}^2 w) \boldsymbol{n}_{TF} \right)_F \\
&\quad - \sum_{F \in \mathcal{F}_T} \left(\cancel{\pi_F^k} v, \mathbf{div}\, \mathbb{A}_T \boldsymbol{\nabla}^2 w \cdot \boldsymbol{n}_{TF} \right)_F \\
&= (\mathbb{A}_T \boldsymbol{\nabla}^2 v, \boldsymbol{\nabla}^2 w)_T
\end{aligned}
$$

Local deflection reconstruction II

- As a result, for $v \in H^2(T)$,

$$\left(\mathbb{A}_T \boldsymbol{\nabla}^2(r_T^{k+2}\underline{\mathrm{I}}_T^k v - v), \boldsymbol{\nabla}^2 w\right)_T = 0 \quad \forall w \in \mathbb{P}^{k+2}(T),$$
$$\pi_T^1(r_T^{k+2}\underline{\mathrm{I}}_T^k v - v) = 0$$
$$\Downarrow$$
$$\boxed{r_T^{k+2} \circ \underline{\mathrm{I}}_T^k = \varpi_T^{k+2}}$$

- Thus, $r_T^{k+2} \circ \underline{\mathrm{I}}_T^k$ has optimal H^s-approximation properties

Global problem I

- For all $T \in \mathcal{T}_h$, we define the local bilinear form $\mathrm{a}_T : \underline{\mathrm{U}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}$ by

$$\boxed{\mathrm{a}_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) \coloneqq (\mathbb{A}_T \boldsymbol{\nabla}^2 r_T^{k+2} \underline{\mathrm{u}}_T, \boldsymbol{\nabla}^2 r_T^{k+2} \underline{\mathrm{v}}_T)_T + \mathrm{s}_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T)}$$

- The stabilization term $\mathrm{s}_T : \underline{\mathrm{U}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}$ is s.t.

$$\begin{aligned}
\mathrm{s}_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) \coloneqq{}& \frac{\mathcal{A}_T^+}{h_T^4} \Big(\pi_T^k (r_T^{k+2} \underline{\mathrm{u}}_T - u_T), \pi_T^k (r_T^{k+2} \underline{\mathrm{v}}_T - v_T)\Big)_T \\
&+ \frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \Big(\boldsymbol{\pi}_F^k (\boldsymbol{\nabla} r_T^{k+2} \underline{\mathrm{u}}_T - \boldsymbol{u}_{\boldsymbol{\nabla},F}), \boldsymbol{\pi}_F^k (\boldsymbol{\nabla} r_T^{k+2} \underline{\mathrm{v}}_T - \boldsymbol{v}_{\boldsymbol{\nabla},F})\Big)_F \\
&+ \frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \Big(\pi_F^k (r_T^{k+2} \underline{\mathrm{u}}_T - u_F), \pi_F^k (r_T^{k+2} \underline{\mathrm{v}}_T - v_F)\Big)_F
\end{aligned}$$

- Polynomial consistency: since $r_T^{k+2} \underline{\mathrm{I}}_T^k v = \varpi_T^{k+2} v = v$ for all $v \in \mathbb{P}^{k+2}(T)$,

$$\mathrm{s}_T(\underline{\mathrm{I}}_T^k v, \underline{\mathrm{w}}_T) = 0 \quad \forall \underline{\mathrm{w}}_T \in \underline{\mathrm{U}}_T^k$$

Global problem II

- Define the following global space with single-valued interface unknowns:

$$\underline{\mathrm{U}}_h^k \coloneqq \left(\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T) \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F)^2 \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right)$$

- A global bilinear form is assembled element-wise:

$$\mathrm{a}_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) \coloneqq \sum_{T \in \mathcal{T}_h} \mathrm{a}_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T)$$

Discrete problem

Find $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_{h,0}^k \coloneqq \{\underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k : v_F = 0,\ \boldsymbol{v}_{\boldsymbol{\nabla},F} = \mathbf{0}$ for any $F \in \mathcal{F}_h^{\mathrm{b}}\}$ s.t.

$$\boxed{\mathrm{a}_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) = (f, v_h)}$$

with $v_{h|T} = v_T$ for all $T \in \mathcal{T}_h$

Global problem III

- Define on $\underline{\mathrm{U}}_{h,0}^{k}$ the following norm

$$\|\underline{\mathrm{v}}_h\|_{\mathbb{A},h} \coloneqq \sum_{T\in\mathcal{T}_h} \left(\|\mathbb{A}_T^{1/2}\boldsymbol{\nabla}^2 v_T\|_T^2 + \frac{\mathcal{A}_T^+}{h_T} \sum_{F\in\mathcal{F}_T} \|\boldsymbol{v}_{\boldsymbol{\nabla},F} - \boldsymbol{\nabla} v_T\|_F^2 + \frac{\mathcal{A}_T^+}{h_T^3} \sum_{F\in\mathcal{F}_T} \|v_F - v_T\|_F^2 \right)^{1/2}$$

- The global bilinear form a_h is coercive and bounded:

$$\boxed{\|\underline{\mathrm{v}}_h\|_{\mathbb{A},h}^2 \lesssim \mathrm{a}_h(\underline{\mathrm{v}}_h, \underline{\mathrm{v}}_h) \lesssim \|\underline{\mathrm{v}}_h\|_{\mathbb{A},h}^2 \qquad \forall \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^{k}}$$

- The global bilinear form a_h is consistent: for all $v \in H^{k+3}(\Omega) \cap H_0^2(\Omega)$,

$$\boxed{\sup_{\underline{\mathrm{w}}_h \in \underline{\mathrm{U}}_{h,0}^{k}\setminus\{\underline{0}_h\}} \frac{(\mathrm{div}\,\mathbf{div}\,\mathbb{A}\boldsymbol{\nabla}^2 v, w_h) - \mathrm{a}_h(\underline{\mathrm{I}}_h^k v, \underline{\mathrm{w}}_h)}{\|\underline{\mathrm{w}}_h\|_{\mathbb{A},h}} \lesssim h^{k+1}|v|_{H^{k+3}(\Omega)}}$$

Energy-error estimate

- Define the global deflection reconstruction $r_h^{k+2}\colon \underline{\mathrm{U}}_h^k \to L^2(\Omega)$ s.t., for all $\underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k$,

$$(r_h^{k+2}\underline{\mathrm{v}}_h)_{|T} = r_T^{k+2}\underline{\mathrm{v}}_T \qquad \forall T \in \mathcal{T}_h$$

- Define the following stabilization seminorm on $\underline{\mathrm{U}}_h^k$

$$|\underline{\mathrm{v}}_h|_{\mathrm{s},h}^2 \coloneqq \sum_{T\in\mathcal{T}_h} \mathrm{s}_T(\underline{\mathrm{v}}_T, \underline{\mathrm{v}}_T)$$

Theorem (Energy-error estimate)

Let $u \in H_0^2(\Omega)$ and $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_{h,0}^k$. Assume the additional regularity $u \in H^{k+3}(\Omega)$. Then, there is $C > 0$ depending on $\mathbb{A}$, but independent of h, s.t.

$$\|\mathbb{A}^{1/2}\boldsymbol{\nabla}_h^2(r_h^{k+2}\underline{\mathrm{u}}_h - u)\| + |\underline{\mathrm{u}}_h|_{\mathrm{s},h} \leqslant Ch^{k+1}|u|_{H^{k+3}(\Omega)}$$

- Choosing $k = 1$ we recover the HCT error estimate

L^2-error estimate

- To infer a sharp L^2-error estimate, we assume biharmonic regularity:

  For all $q \in L^2(\Omega)$, the solution $z \in H_0^2(\Omega)$ to

  $$(\mathbb{A}\boldsymbol{\nabla}^2 z, \boldsymbol{\nabla}^2 v) = (q, v) \qquad \forall v \in H_0^2(\Omega)$$

  satisfies the *a priori* estimate

  $$\|z\|_{H^4(\Omega)} \leqslant C_{\text{bihar}}\|q\|,$$

  with $C_{\text{bihar}} > 0$ only depending on Ω

Theorem (L^2-error estimate)

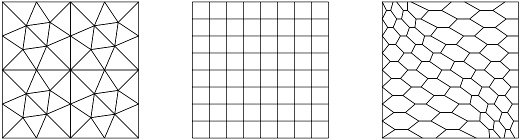
Let $u \in H_0^2(\Omega)$ and $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_{h,0}^k$. Assume biharmonic regularity, $f \in H^{k+1}(\mathcal{T}_h)$, and $u \in H^{k+3}(\Omega)$. Then, there is $C > 0$ depending on $\mathbb{A}$, but independent of h, s.t.

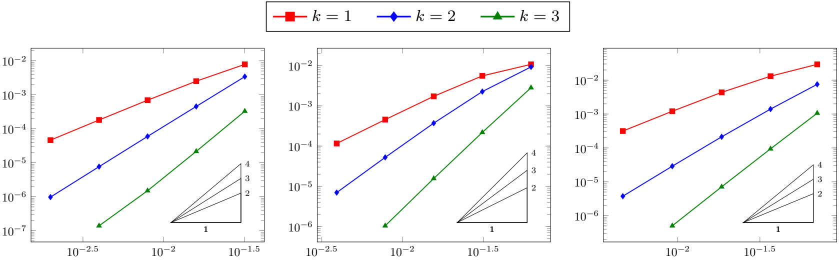
$$\|r_h^{k+2}\underline{\mathrm{u}}_h - u\| \leqslant Ch^{k+3}\left(\|u\|_{H^{k+3}(\Omega)} + \|f\|_{H^{k+1}(\mathcal{T}_h)}\right)$$

Numerical examples I

- We solve the biharmonic equation on $\Omega = (0,1) \times (0,1)$, for

$$u(x,y) = x^2(1-x)^2y^2(1-y)^2$$





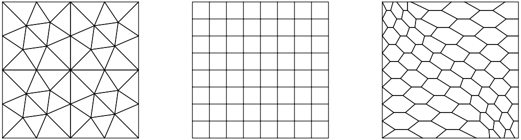
(a) Triangular (b) Cartesian (c) Hexagonal

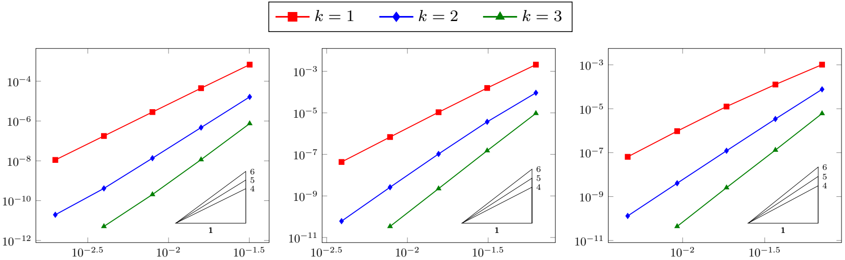
Figure: Energy error vs. meshsize for three different meshes

Numerical examples II

- We solve the biharmonic equation on $\Omega = (0,1) \times (0,1)$, for

$$u(x,y) = x^2(1-x)^2 y^2 (1-y)^2$$





(a) Triangular (b) Cartesian (c) Hexagonal

Figure: L^2-error vs. meshsize for three different meshes

Conclusions & perspectives

Conclusions

- We presented a new nonconforming method based on a **primal** formulation
- Choosing $k = 1$ is enough to get a **quadratic** energy error and a **quartic** L^2-error
- **Mechanical equilibrium principles** are reproduced at the **discrete level**

Perspectives

- Consider the case of **simply supported plates**:

$$u = 0 \quad \text{and} \quad (\mathbb{A}\boldsymbol{\nabla}^2 u)\boldsymbol{n} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega$$

- Consider a variant based on a **dual** or **mixed** formulation
- Consider the case of **Reissner–Mindlin** plates

Thanks for your attention

Discrete PVW & Laws of action-reaction I

- Let $T \in \mathcal{T}_h$ be fixed, and $\boldsymbol{M}_T \coloneqq -\mathbb{A}_T \boldsymbol{\nabla}^2 u$
- At the **continuous** level
  - The **principle of virtual work** holds:

$$\text{For all } v \in \mathbb{P}^k(T),$$

$$-(\boldsymbol{M}_T, \boldsymbol{\nabla}^2 v)_T + \sum_{F \in \mathcal{F}_T} (\boldsymbol{M}_T \boldsymbol{n}_{TF}, \boldsymbol{\nabla} v)_F - \sum_{F \in \mathcal{F}_T} (\mathbf{div}\, \boldsymbol{M}_T \cdot \boldsymbol{n}_{TF}, v)_F = (f, v)_T$$

  - The following **laws of action-reaction** hold, for $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$:

$$\boldsymbol{M}_{T_1} \boldsymbol{n}_{T_1 F} + \boldsymbol{M}_{T_2} \boldsymbol{n}_{T_2 F} = \boldsymbol{0}, \qquad \mathbf{div}\, \boldsymbol{M}_{T_1} \cdot \boldsymbol{n}_{T_1 F} + \mathbf{div}\, \boldsymbol{M}_{T_2} \cdot \boldsymbol{n}_{T_2 F} = 0$$

- The solution to the discrete problem satisfies discrete counterparts of the above statements

Discrete PVW & Laws of action-reaction II

- Define the space

$$\underline{\mathrm{D}}_{\partial T}^k \coloneqq \left(\bigtimes_{F\in\mathcal{F}_T} \mathbb{P}^k(F)^2 \right) \times \left(\bigtimes_{F\in\mathcal{F}_T} \mathbb{P}^k(F) \right)$$

and the boundary difference operator $\underline{\delta}_{\partial T}^k : \underline{\mathrm{U}}_T^k \to \underline{\mathrm{D}}_{\partial T}^k$ s.t., for all $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$,

$$\begin{aligned}\underline{\delta}_{\partial T}^k \underline{\mathrm{v}}_T &\equiv \left((\boldsymbol{\delta}_{\boldsymbol{\nabla},F}^k \underline{\mathrm{v}}_T)_{F\in\mathcal{F}_T}, (\delta_F^k \underline{\mathrm{v}}_T)_{F\in\mathcal{F}_T} \right) \\ &\coloneqq \left((\boldsymbol{v}_{\boldsymbol{\nabla},F} - \boldsymbol{\nabla} v_T)_{F\in\mathcal{F}_T}, (v_F - v_T)_{F\in\mathcal{F}_T} \right)\end{aligned}$$

- Define now the residual operator

$$\underline{\mathrm{R}}_{\partial T}^k \equiv \left((\boldsymbol{R}_{\boldsymbol{\nabla},F}^k)_{F\in\mathcal{F}_T}, (R_F^k)_{F\in\mathcal{F}_T} \right) : \underline{\mathrm{U}}_T^k \to \underline{\mathrm{D}}_{\partial T}^k$$

s.t., for all $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$ and all $\underline{\alpha}_{\partial T} \equiv ((\boldsymbol{\alpha}_{\boldsymbol{\nabla},F})_{F\in\mathcal{F}_T}, (\alpha_F)_{F\in\mathcal{F}_T}) \in \underline{\mathrm{D}}_{\partial T}^k$,

$$\begin{aligned}(\underline{\mathrm{R}}_{\partial T}^k \underline{\mathrm{v}}_T, \underline{\alpha}_{\partial T})_{0,\partial T} &\equiv \sum_{F\in\mathcal{F}_T} \left((\boldsymbol{R}_{\boldsymbol{\nabla},F}^k \underline{\mathrm{v}}_T, \boldsymbol{\alpha}_{\boldsymbol{\nabla},F})_F + (R_F^k \underline{\mathrm{v}}_T, \alpha_F)_F \right) \\ &= \mathrm{s}_T((0, \underline{\delta}_{\partial T}^k \underline{\mathrm{v}}_T), (0, \underline{\alpha}_{\partial T}))\end{aligned}$$

Discrete PVW & Laws of action-reaction III

Lemma (Local principle of virtual work and laws of action-reaction)

Let $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_{h,0}^k$ be the unique solution to the discrete problem and, for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, define the **discrete moment and shear force**

$$\boxed{\begin{aligned} \boldsymbol{\mathcal{M}}_{TF}^k(\underline{\mathrm{u}}_T) &:= -\left((\mathbb{A}\boldsymbol{\nabla}^2 r_T^{k+2}\underline{\mathrm{u}}_T)\boldsymbol{n}_{TF} + \boldsymbol{R}_{\boldsymbol{\nabla},F}^k\underline{\mathrm{u}}_T\right), \\ \mathcal{S}_{TF}^k(\underline{\mathrm{u}}_T) &:= -\,\mathbf{div}\,\mathbb{A}\boldsymbol{\nabla}^2 r_T^{k+2}\underline{\mathrm{u}}_T \cdot \boldsymbol{n}_{TF} + R_F^k\underline{\mathrm{u}}_T. \end{aligned}}$$

Then, the following **discrete counterparts** of PVW and laws of action-reaction hold, respectively:

For any mesh element $T \in \mathcal{T}_h$, and for all $v_T \in \mathbb{P}^k(T)$,

$$(\mathbb{A}_T\boldsymbol{\nabla}^2 r_T^{k+2}\underline{\mathrm{u}}_T, v_T)_T + \sum_{F\in\mathcal{F}_T}(\boldsymbol{\mathcal{M}}_{TF}^k(\underline{\mathrm{u}}_T), \boldsymbol{\nabla}v_T)_F - \sum_{F\in\mathcal{F}_T}(\mathcal{S}_{TF}^k(\underline{\mathrm{u}}_T), v_T)_F = (f, v_T)_T;$$

For any interface $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$,

$$\boldsymbol{\mathcal{M}}_{T_1F}^k(\underline{\mathrm{u}}_{T_1}) + \boldsymbol{\mathcal{M}}_{T_2F}^k(\underline{\mathrm{u}}_{T_2}) = \mathbf{0}, \qquad \mathcal{S}_{T_1F}^k(\underline{\mathrm{u}}_{T_1}) + \mathcal{S}_{T_2F}^k(\underline{\mathrm{u}}_{T_2}) = 0.$$