A Hybrid High-Order method for Kirchhoff–Love plate bending problems

Francesco Bonaldi

joint work with D. A. Di Pietro, G. Geymonat, and F. Krasucki

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Outline

Motivations

- Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected polygonal 2D domain

K–L clamped plate bending problem

For $f \in L^2(\Omega)$, find $u \in H_0^2(\Omega)$ such that

$$(\mathbb{A}\nabla^2 u, \nabla^2 v) = (f, v), \ \forall v \in H_0^2(\Omega)$$

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- A conforming discretization is computationally expensive (C^1 elements)
- HCT energy-error estimate [Ciarlet, 1974]: provided $u \in H^4(\Omega)$,

$$\|u - u_h\|_{H^2(\Omega)} \leqslant Ch^2 |u|_{H^4(\Omega)}$$

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$$\Downarrow$$

- **Goal**: devising a new nonconforming numerical scheme, so as to improve the computational cost of the method

Key ideas for HHO

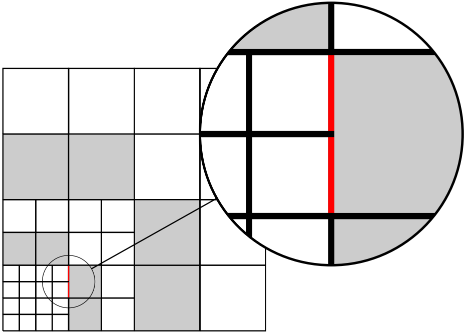
- Discrete unknowns
  - Polynomials of order $k \geqslant 1$ on mesh cells and faces
  - Cell-based unknowns can be eliminated by static condensation
- Building principles
  - Reconstruction operator based on local primal Neumann problems
  - Face-based penalty linking cell- and face-based unknowns
- Main benefits
  - Capability of handling general polygonal meshes
  - High-order method: energy-error estimate of order $(k+1)$ and L^2-error estimate of order $(k+3)$ for smooth solutions
  - Reproduction of key continuous mechanical properties at the discrete level

Mesh

Mesh regularity

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$, and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the usual sense of Ciarlet
- contact-regular, i.e., every simplex $S \subset T$ is s.t. $h_S \approx h_T$



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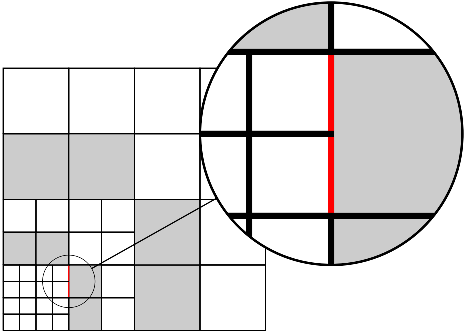
Consequences [Di Pietro & Ern, 2012; Di Pietro & Droniou, 2017]:

- L^2-trace and inverse inequalities
- Approximation for broken polynomial spaces

Hypothesis: the material tensor field $\mathbb{A}$ is **element-wise constant**; we set

$$\mathbb{A}_T := \mathbb{A}_{|T} \quad \forall T \in \mathcal{T}_h$$



Projectors on local polynomial spaces

- The L^2-orthogonal projector $\pi_X^\ell \colon L^2(X) \to \mathbb{P}^\ell(X)$ is s.t.

$$(\pi_X^\ell v - v, w)_X = 0 \quad \forall w \in \mathbb{P}^\ell(X)$$

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- The local energy projector $\varpi_T^\ell \colon H^2(T) \to \mathbb{P}^\ell(T)$, for $\ell \geqslant 2$, is s.t.

$$\left(\mathbb{A}_T \boldsymbol{\nabla}^2(\varpi_T^\ell v - v), \boldsymbol{\nabla}^2 w\right)_T = 0 \quad \forall w \in \mathbb{P}^\ell(T),$$
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- Both projectors have **optimal approximation properties** in $H^s(T)$

Theorem (Optimal approximation properties of ϖ_T^ℓ)

There is $C > 0$ independent of h, but possibly depending on $\mathbb{A}$, s.t., for all $T \in \mathcal{T}_h$, all $s \in \{2, \ldots, \ell+1\}$, and all $v \in H^s(T)$,

$$|v - \varpi_T^\ell v|_{H^m(T)} \leqslant C h_T^{s-m} |v|_{H^s(T)} \qquad \forall m \in \{0, \ldots, s-1\},$$
$$|v - \varpi_T^\ell v|_{H^m(\partial T)} \leqslant C h_T^{s-m-1/2} |v|_{H^s(T)} \qquad \forall m \in \{0, \ldots, s-1\}.$$

Local unknowns and interpolation

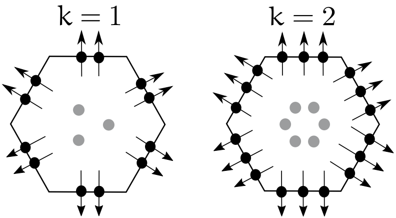

Figure: $\underline{U}_T^k$ for $k \in \{1, 2\}$

- For $k \geqslant 1$, and $T \in \mathcal{T}_h$, we define the local space of discrete unknowns

$$\boxed{\underline{U}_T^k := \mathbb{P}^k(T) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F)^2 \right) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right)}$$

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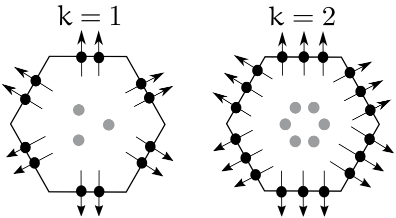

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- The local interpolator $\underline{I}_T^k \colon H^2(T) \to \underline{U}_T^k$ is s.t.

$$\underline{I}_T^k v := (\pi_T^k v, (\boldsymbol{\pi}_F^k \left((\boldsymbol{\nabla} v)_{|F} \right))_{F \in \mathcal{F}_T}, (\pi_F^k (v_{|F}))_{F \in \mathcal{F}_T})$$

Local deflection reconstruction I

- We define the local deflection reconstruction $\mathrm{p}_T^{k+2}\colon \underline{U}_T^k \to \mathbb{P}^{k+2}(T)$ s.t.

$$
\begin{aligned}
(\mathbb{A}_T \nabla^2 \mathrm{p}_T^{k+2} \underline{v}_T, \nabla^2 w)_T = {} & (\mathbb{A}_T \nabla^2 v_T, \nabla^2 w)_T \\
& + \sum_{F \in \mathcal{F}_T} \left(\boldsymbol{v}_{\nabla,F} - \nabla v_T, (\mathbb{A}_T \nabla^2 w) \mathbf{n}_{TF} \right)_F \\
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for all $w \in \mathbb{P}^{k+2}(T)$

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for all $w \in \mathbb{P}^{k+2}(T)$, with **closure condition**

$$\pi_T^1(\mathrm{p}_T^{k+2}\underline{v}_T - v_T) = 0$$

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\end{aligned}
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$$

- An integration by parts on the first term on the right-hand side yields

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\begin{aligned}
(\mathbb{A}_T \nabla^2 \mathrm{p}_T^{k+2}\underline{v}_T, \nabla^2 w)_T &= (v_T, \operatorname{div}\mathbf{div}\,\mathbb{A}_T \nabla^2 w)_T \\
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Local deflection reconstruction II

- By the definition of $\underline{I}_T^k$ it holds, for all $v \in H^2(T)$ and all $w \in \mathbb{P}^{k+2}(T)$,

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(\mathbb{A}_T \boldsymbol{\nabla}^2 p_T^{k+2} \underline{I}_T^k v, \boldsymbol{\nabla}^2 w)_T = &(\pi_T^k v, \operatorname{div} \mathbf{div}\, \mathbb{A}_T \boldsymbol{\nabla}^2 w)_T \\
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- As a result, for $v \in H^2(T)$,

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\pi_T^1 (\mathrm{p}_T^{k+2} \underline{\mathrm{I}}_T^k v - v) = 0 \\
\Downarrow \\
\boxed{\mathrm{p}_T^{k+2} \circ \underline{\mathrm{I}}_T^k = \varpi_T^{k+2}}
\end{gathered}$$

- Thus, $\mathrm{p}_T^{k+2} \circ \underline{\mathrm{I}}_T^k$ has optimal H^s-approximation properties

Global problem I

- For all $T \in \mathcal{T}_h$, we define the local bilinear form $a_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ by

$$\boxed{a_T(\underline{u}_T, \underline{v}_T) := (\mathbb{A}_T \nabla^2 p_T^{k+2} \underline{u}_T, \nabla^2 p_T^{k+2} \underline{v}_T)_T + s_T(\underline{u}_T, \underline{v}_T)}$$

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- The stabilization term $s_T \colon \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ is s.t.

$$\begin{aligned} s_T(\underline{u}_T, \underline{v}_T) \coloneqq {} & \frac{\mathcal{A}_T^+}{h_T^4} \Big(\pi_T^k (p_T^{k+2} \underline{u}_T - u_T), \pi_T^k (p_T^{k+2} \underline{v}_T - v_T) \Big)_T \\ & + \frac{\mathcal{A}_T^+}{h_T} \sum_{F \in \mathcal{F}_T} \Big(\boldsymbol{\pi}_F^k (\boldsymbol{\nabla} p_T^{k+2} \underline{u}_T - \boldsymbol{u}_{\boldsymbol{\nabla},F}), \boldsymbol{\pi}_F^k (\boldsymbol{\nabla} p_T^{k+2} \underline{v}_T - \boldsymbol{v}_{\boldsymbol{\nabla},F}) \Big)_F \\ & + \frac{\mathcal{A}_T^+}{h_T^3} \sum_{F \in \mathcal{F}_T} \Big(\pi_F^k (p_T^{k+2} \underline{u}_T - u_F), \pi_F^k (p_T^{k+2} \underline{v}_T - v_F) \Big)_F \end{aligned}$$

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\end{aligned}$$

- Polynomial consistency: since $p_T^{k+2}\underline{I}_T^k v = \varpi_T^{k+2} v = v$ for all $v \in \mathbb{P}^{k+2}(T)$,

$$s_T(\underline{I}_T^k v, \underline{w}_T) = 0 \quad \forall \underline{w}_T \in \underline{U}_T^k$$

Global problem II

- Define the following global space with single-valued interface unknowns:

$$\underline{U}_h^k := \left(\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T) \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F)^2 \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right)$$

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- A global bilinear form is assembled element-wise:

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Discrete problem

Find $\underline{u}_h \in \underline{U}_{h,0}^k := \{\underline{v}_h \in \underline{U}_h^k : v_F = 0,\ \boldsymbol{v}_{\nabla,F} = 0 \text{ for any } F \in \mathcal{F}_h^b\}$ s.t.

$$\boxed{a_h(\underline{u}_h, \underline{v}_h) = (f, v_h)}$$

with $v_{h|T} = v_T$ for all $T \in \mathcal{T}_h$

Global problem III

- Define on $\underline{U}_{h,0}^{k}$ the following norm

$$\|\underline{v}_h\|_{\mathbb{A},h} := \sum_{T\in\mathcal{T}_h}\left(\|\mathbb{A}_T^{1/2}\boldsymbol{\nabla}^2 v_T\|_T^2 + \frac{\mathcal{A}_T^+}{h_T}\sum_{F\in\mathcal{F}_T}\|\boldsymbol{v}_{\boldsymbol{\nabla},F} - \boldsymbol{\nabla} v_T\|_F^2 + \frac{\mathcal{A}_T^+}{h_T^3}\sum_{F\in\mathcal{F}_T}\|v_F - v_T\|_F^2\right)^{1/2}$$

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$$\|\underline{v}_h\|_{\mathbb{A},h} := \sum_{T\in\mathcal{T}_h}\left(\|\mathbb{A}_T^{1/2}\nabla^2 v_T\|_T^2 + \frac{\mathcal{A}_T^+}{h_T}\sum_{F\in\mathcal{F}_T}\|v_{\nabla,F}-\nabla v_T\|_F^2 + \frac{\mathcal{A}_T^+}{h_T^3}\sum_{F\in\mathcal{F}_T}\|v_F - v_T\|_F^2\right)^{1/2}$$

- The global bilinear form a_h is coercive and bounded:

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- The global bilinear form a_h is consistent: for all $v \in H^{k+3}(\Omega)\cap H_0^2(\Omega)$,

$$\boxed{\sup_{\underline{w}_h\in\underline{U}_{h,0}^{k}\setminus\{\underline{0}_h\}}\frac{(\operatorname{div}\mathbf{div}\,\mathbb{A}\boldsymbol{\nabla}^2 v, w_h) - a_h(\underline{I}_h^k v,\underline{w}_h)}{\|\underline{w}_h\|_{\mathbb{A},h}} \lesssim h^{k+1}|v|_{H^{k+3}(\Omega)}}$$

Energy-error estimate

- Define the global deflection reconstruction $p_h^{k+2}\colon \underline{U}_h^k \to L^2(\Omega)$ s.t., for all $\underline{v}_h \in \underline{U}_h^k$,
$$(p_h^{k+2}\underline{v}_h)_{|T} = p_T^{k+2}\underline{v}_T \qquad \forall T \in \mathcal{T}_h$$
- Define the following stabilization seminorm on $\underline{U}_h^k$
$$|\underline{v}_h|_{s,h}^2 := \sum_{T\in\mathcal{T}_h} s_T(\underline{v}_T, \underline{v}_T)$$

Theorem (Energy-error estimate)

Let $u \in H_0^2(\Omega)$ and $\underline{u}_h \in \underline{U}_{h,0}^k$. Assume the additional regularity $u \in H^{k+3}(\Omega)$. Then, there is $C > 0$ depending on $\mathbb{A}$, but independent of h, s.t.

$$\|\mathbb{A}^{1/2}\boldsymbol{\nabla}_h^2(p_h^{k+2}\underline{u}_h - u)\| + |\underline{u}_h|_{s,h} \leqslant Ch^{k+1}|u|_{H^{k+3}(\Omega)}.$$

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- Choosing $k = 1$ we recover the HCT error estimate

L^2-error estimate

- To infer a sharp L^2-error estimate, we assume biharmonic regularity:

  For all $q \in L^2(\Omega)$, the solution $z \in H_0^2(\Omega)$ to

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Theorem (L^2-error estimate)

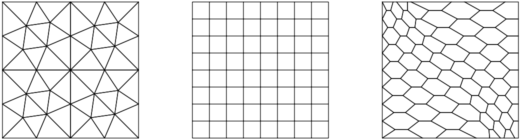
Let $u \in H^2_0(\Omega)$ and $\underline{u}_h \in \underline{U}^k_{h,0}$. Assume biharmonic regularity, $f \in H^{k+1}(\mathcal{T}_h)$, and $u \in H^{k+3}(\Omega)$. Then, there is $C > 0$ depending on $\mathbb{A}$, but independent of h, s.t.

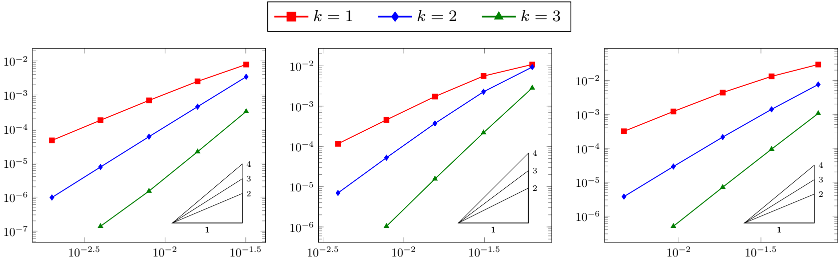
$$\|p_h^{k+2}\underline{u}_h - u\| \leqslant Ch^{k+3}\left(\|u\|_{H^{k+3}(\Omega)} + \|f\|_{H^{k+1}(\mathcal{T}_h)}\right).$$

Numerical examples I

- We solve the biharmonic equation on $\Omega = (0,1) \times (0,1)$, for

$$u(x,y) = x^2(1-x)^2y^2(1-y)^2$$





(a) Triangular (b) Cartesian (c) Hexagonal

Figure: Energy error vs. meshsize for three different meshes

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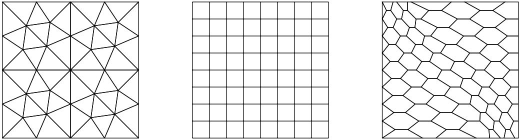

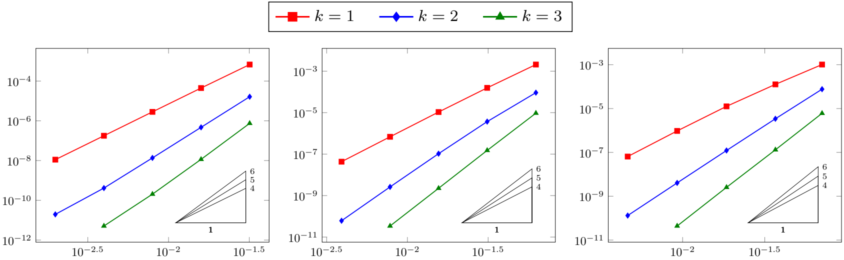

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Figure: L^2-error vs. meshsize for three different meshes

Discrete PVW & Laws of action-reaction I

- Let $T \in \mathcal{T}_h$ be fixed, and $\mathbf{M}_T \coloneqq -\mathbb{A}_T \boldsymbol{\nabla}^2 u$

Discrete PVW & Laws of action-reaction I

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- At the **continuous** level
  - The **principle of virtual work** holds:

$$\text{For all } v \in \mathbb{P}^k(T),$$

$$-(\mathbf{M}_T, \boldsymbol{\nabla}^2 v)_T + \sum_{F \in \mathcal{F}_T} (\mathbf{M}_T \mathbf{n}_{TF}, \boldsymbol{\nabla} v)_F - \sum_{F \in \mathcal{F}_T} (\mathbf{div}\, \mathbf{M}_T \cdot \mathbf{n}_{TF}, v)_F = (f, v)_T$$

  - The following **laws of action-reaction** hold, for $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$:

$$\mathbf{M}_{T_1} \mathbf{n}_{T_1 F} + \mathbf{M}_{T_2} \mathbf{n}_{T_2 F} = 0, \qquad \mathbf{div}\, \mathbf{M}_{T_1} \cdot \mathbf{n}_{T_1 F} + \mathbf{div}\, \mathbf{M}_{T_2} \cdot \mathbf{n}_{T_2 F} = 0$$

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- The solution to the discrete problem satisfies discrete counterparts of the above statements

Discrete PVW & Laws of action-reaction II

- Define the space

$$\underline{D}_{\partial T}^{k} \coloneqq \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F)^2 \right) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right)$$

and the boundary difference operator $\underline{\delta}_{\partial T}^{k} : \underline{U}_T^k \to \underline{D}_{\partial T}^{k}$ s.t., for all $\underline{v}_T \in \underline{U}_T^k$,

$$\begin{aligned}\underline{\delta}_{\partial T}^{k} \underline{v}_T &\equiv \left((\boldsymbol{\delta}_{\boldsymbol{\nabla},F}^{k} \underline{v}_T)_{F \in \mathcal{F}_T}, (\delta_F^k \underline{v}_T)_{F \in \mathcal{F}_T} \right) \\ &\coloneqq \left((\boldsymbol{v}_{\boldsymbol{\nabla},F} - \boldsymbol{\nabla} v_T)_{F \in \mathcal{F}_T}, (v_F - v_T)_{F \in \mathcal{F}_T} \right)\end{aligned}$$

Discrete PVW & Laws of action-reaction II

- Define the space

$$\underline{D}^k_{\partial T} := \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F)^2 \right) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right)$$

and the boundary difference operator $\underline{\delta}^k_{\partial T} : \underline{U}^k_T \to \underline{D}^k_{\partial T}$ s.t., for all $\underline{v}_T \in \underline{U}^k_T$,

$$\begin{aligned} \underline{\delta}^k_{\partial T}\underline{v}_T &\equiv \left((\boldsymbol{\delta}^k_{\nabla,F}\underline{v}_T)_{F \in \mathcal{F}_T}, (\delta^k_F \underline{v}_T)_{F \in \mathcal{F}_T} \right) \\ &:= \left((\boldsymbol{v}_{\nabla,F} - \boldsymbol{\nabla} v_T)_{F \in \mathcal{F}_T}, (v_F - v_T)_{F \in \mathcal{F}_T} \right) \end{aligned}$$

- Define now the residual operator

$$\underline{R}^k_{\partial T} \equiv \left((\boldsymbol{R}^k_{\nabla,F})_{F \in \mathcal{F}_T}, (R^k_F)_{F \in \mathcal{F}_T} \right) : \underline{U}^k_T \to \underline{D}^k_{\partial T}$$

s.t., for all $\underline{v}_T \in \underline{U}^k_T$ and all $\underline{\alpha}_{\partial T} \equiv \left((\boldsymbol{\alpha}_{\nabla,F})_{F \in \mathcal{F}_T}, (\alpha_F)_{F \in \mathcal{F}_T} \right) \in \underline{D}^k_{\partial T}$,

$$\begin{aligned} (\underline{R}^k_{\partial T}\underline{v}_T, \underline{\alpha}_{\partial T})_{0,\partial T} &\equiv \sum_{F \in \mathcal{F}_T} \left((\boldsymbol{R}^k_{\nabla,F}\underline{v}_T, \boldsymbol{\alpha}_{\nabla,F})_F + (R^k_F \underline{v}_T, \alpha_F)_F \right) \\ &= s_T((0, \underline{\delta}^k_{\partial T}\underline{v}_T), (0, \underline{\alpha}_{\partial T})) \end{aligned}$$

Discrete PVW & Laws of action-reaction III

Lemma (Local principle of virtual work and laws of action-reaction)

Let $\underline{u}_h \in \underline{U}^k_{h,0}$ be the unique solution to the discrete problem and, for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, define the **discrete moment and shear force**

$$\boxed{\begin{aligned} \mathcal{M}^k_{TF}(\underline{u}_T) &:= -\big((\mathbb{A}\boldsymbol{\nabla}^2 p^{k+2}_T \underline{u}_T)\boldsymbol{n}_{TF} + \mathbf{R}^k_{\boldsymbol{\nabla},F}\underline{u}_T\big), \\ \mathcal{S}^k_{TF}(\underline{u}_T) &:= -\,\mathbf{div}\,\mathbb{A}\boldsymbol{\nabla}^2 p^{k+2}_T \underline{u}_T \cdot \boldsymbol{n}_{TF} + R^k_F \underline{u}_T. \end{aligned}}$$

Then, the following **discrete counterparts** of PVW and laws of action-reaction hold, respectively:

For any mesh element $T \in \mathcal{T}_h$, and for all $v_T \in \mathbb{P}^k(T)$,

$$(\mathbb{A}_T \boldsymbol{\nabla}^2 p^{k+2}_T \underline{u}_T, v_T)_T + \sum_{F \in \mathcal{F}_T} (\mathcal{M}^k_{TF}(\underline{u}_T), \boldsymbol{\nabla} v_T)_F - \sum_{F \in \mathcal{F}_T} (\mathcal{S}^k_{TF}(\underline{u}_T), v_T)_F = (f, v_T)_T;$$

For any interface $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$,

$$\mathcal{M}^k_{T_1F}(\underline{u}_{T_1}) + \mathcal{M}^k_{T_2F}(\underline{u}_{T_2}) = 0, \qquad \mathcal{S}^k_{T_1F}(\underline{u}_{T_1}) + \mathcal{S}^k_{T_2F}(\underline{u}_{T_2}) = 0.$$

Conclusions & perspectives

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- We presented a new nonconforming method based on a **primal** formulation
- Choosing $k = 1$ is enough to get a **quadratic** energy error and a **quartic** L^2-error
- **Mechanical equilibrium principles** are reproduced at the **discrete level**

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Perspectives

- Consider the case of **simply supported plates**:

$$u = 0 \quad \text{and} \quad (\mathbb{A}\boldsymbol{\nabla}^2 u)\mathbf{n} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega$$

- Consider a variant based on a **dual** formulation
- Couple the method with a **time discretization** to treat dynamics of plates

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