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Modélisation Mathématique et Numérique de Structures en présence de Couplages Linéaires Multiphysiques

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Introduction

Cette thèse est consacrée à l'enrichissement du modèle mathématique classique des *structures intelligentes*, en tenant compte des effets thermiques, et à son étude analytique et numérique. Par cette expression, on se réfère typiquement à des structures se présentant sous forme de *capteurs* ou *actionneurs*, piézoélectriques et/ou magnétostrictifs, avec une géométrie de plaque ou de coque. L'influence de la température se traduit par des couplages supplémentaires, donnant lieu aux phénomènes de *thermoélasticité* (couplage thermo-mécanique), *pyroélectricité* (couplage thermo-électrique) et *pyromagnétisme* (couplage thermo-magnétique).

Le présent manuscrit est divisé en quatre parties. Dans la première partie, nous présentons les concepts mathématiques et mécaniques fondamentaux utilisés dans ce travail. La deuxième partie est consacrée au traitement analytique des problèmes mathématiques rencontrés, posés dans un domaine tridimensionnel, et à la déduction de modèles bidimensionnels grâce à la méthode des développements asymptotiques en considérant un domaine en forme de plaque dont l'épaisseur tend vers zéro. Dans la troisième partie, on focalisera notre attention sur le traitement numérique d'un problème de plaque en flexion, qui se présente sous la même forme dans tous les modèles bidimensionnels déduits dans la partie précedente et qui tient compte d'un effet d'inertie de rotation. Enfin, dans la quatrième partie, nous présentons une première approche numérique du problème de plaque en flexion en utilisant une méthode de discrétisation non conforme.

Nous considérons dans ce document les matériaux *magnéto-électro-thermo-élastiques* (METE) comme matériaux intelligents ; l'exemple le plus classique est donné par un composite $BaTiO_3$-$CoFe_2O_4$ (titanate de baryum et ferrite de cobalt). Le couplage mécanique entre les composantes piézoélectrique et magnétostrictive dans une structure faite d'un tel matériau donne lieu à l'ainsi dit effet *magnétoélectrique*, qui n'est pas présent dans les composantes individuelles. Cet effet se manifeste, par exemple, dans des composites multi-couche [39, 40], dans des structures faites d'une matrice homogène dans laquelle des particules de formes différentes (pour la plupart ellipsoïdales) sont dispersées [28, 29], ou bien dans des structures fibreuses, où des cylindres parallèles sont insérés dans la matrice homogène [4]. De plus, dans la plupart de la littérature on considère un modèle *linéaire* : dans ce contexte, le couplage magnéto-élastique étant exprimé par une loi de comportement linéaire entre la contrainte et le champ magnétique, le mot "magnétostrictif" est remplacé par *piézomagnétique* ; nous suivrons la même approche et adopterons la même convention. Dans certains cas, l'influence de la température sur de telles structures ne peut pas être négligée : en effet, la pyroélectricité et le pyromagnétisme peuvent être importants en ce qui concerne les performances de récolte d'énergie (voir, e.g., Kim et al. [40]). Il faut alors rajouter l'équation provenant du bilan de l'énergie au modèle usuel. Pour une description détaillée des couplages et des phénomènes multiphysiques ayant lieu dans ces structures, ainsi que de leurs applications, nous renvoyons aux références [2], [12], [35], [36], [42].

Une caractéristique distinctive des problèmes rencontrés dans les applications est la présence de plusieurs paramètres, ce qui montre la coexistence de différentes échelles : par exemple, l'épaisseur d'une couche piezoélectrique/piézomagnétique peut être petite par rapport aux autres dimensions de la structure, l'influence de la température peut être importante seulement sur certaines inconnues, etc. La superposition de deux phénomènes de propagation d'ondes caractérisés par des vitesses complètement différentes, comme dans le cas des ondes élastiques et électromagnétiques, entraîne un traitement numérique impraticable du problème. Cette question peut se résoudre en ayant recours au modèle *quasi-statique* – où l'on désigne par cette expression l'hypothèse que *les champs électrique et magnétique puissent s'exprimer comme gradients des potentiels correspondants* – qui est justifié par une procédure d'adimensionnalisation sur les équations du problème. L'adimensionnalisation met en évidence un *petit paramètre* δ, i.e., le rapport entre la vitesse maximale de propagation d'une onde élastique dans le milieu et la vitesse de la lumière. Une telle procédure a été effectuée dans [38] sans considérer les effets de la température ; une hypothèse quasi-statique *a priori* a été faite dans [5] et [45] dans le cas d'un matériau thermo-piézoélectrique, dans [1] et [4] dans le cas d'un matériau METE.

Dans ce manuscrit, après la présentation du modèle mathématique et la justification de sa cohérence au sens de la thermodynamique des milieux continus, nous montrons d'abord [9] que le système d'équations aux dérivées partielles qui régit le problème dans sa formulation la plus générale (auquel on se référera dans la suite comme *problème dynamique*) est bien posé dans des espaces fonctionnels opportuns. Pour cela, nous travaillons dans le cadre de la théorie de Hille-Yosida. Une adimensionnalisation des équations du problème dynamique est alors effectuée, de telle façon à (*i*) étendre les résultats d'Imperiale et Joly [38] et (*ii*) justifier l'hypothèse quasi-statique utilisée dans [1], [4], [5] et [45]. Ensuite, nous montrons que le problème quasi-statique est aussi bien posé, grâce à la méthode de Faedo-Galerkin, en suivant l'approche de Lions [43], voir e.g. [45] pour ce type de problèmes.

Dans le chapitre suivant, nous nous intéressons au problème de la modélisation d'une plaque constituée d'un matériau METE dans le cas quasi-statique. Plus précisément, on considère un domaine d'épaisseur h^ε, ε étant un petit paramètre adimensionnel tendant vers zéro. Nous avons recours à la méthode des développements asymptotiques en puissances du petit paramètre afin d'obtenir un modèle bidimensionnel de plaque.

Dans ce contexte, notre approche est semblable à celle présentée par Sène [57], où l'on considère les cas d'un milieu piézoélectrique en régime purement statique (où toutes les variations temporelles sont négligées) et en régime dynamique (où l'on considère, en plus du bilan de la quantité de mouvement, le système complet des équations de Maxwell). En particulier, le même auteur a présenté séparément l'étude du cas statique dans [58] et l'étude du cas dynamique dans un article successif en collaboration avec A. Raoult [53]. Dans ce dernier travail, des effets magnétiques sont pris en compte, en exprimant – en plus des lois de comportement caractérisant un milieu piézoélectrique – le champ d'induction magnétique **B** en fonction du champ magnétique **H** par la relation usuelle $\mathbf{B} = \mu\mathbf{H}$, μ étant la perméabilité magnétique (scalaire) du milieu ; en revanche, les effets de la température sont négligés.

Nous considérons ici quatre types différents de conditions aux limites portant sur les inconnues électromagnétiques, chacun visant à modéliser le comportement de la plaque comme capteur ou actionneur, de nature piézoélectrique ou piézomagnétique [64]. On obtient par conséquent quatre modèles de plaque : le modèle *capteur-actionneur*, selon lequel la plaque se comporte comme un capteur piézoélectrique et un actionneur piézomagnétique ; le modèle *actionneur-capteur*, se référant à un comportement de capteur piézomagnétique et actionneur piézoélectrique ; le modèle *actionneur*, auquel cas on a un comportement d'actionneur à la fois piézoélectrique et piézomagnétique ; et enfin, le modèle *capteur*, se référant à un comportement de capteur à la fois piézoélectrique et piézomagnétique. Les modèles déduits grâce à la méthode des développements asymptotiques sont validés en montrant des résultats de convergence faible lorsque $\varepsilon \to 0$ de la solution du problème de départ vers la solution du problème formulé sur le domaine bidimensionnel. Les quatre modèles bidimensionnels sont déduits en considérant des hypothèses de scaling différentes sur les potentiels électrique et magnétique [63]. En revanche, ils présentent tous des caractéristiques communes : en premier lieu, le champs de déplacement est toujours de type Kirchhoff-Love ; en second lieu, la variation de température est toujours indépendante de la coordonnée d'épaisseur ; enfin, chaque problème se découple en un problème de flexion – régissant l'évolution du déplacement transversal de la plaque et tenant en compte un effet d'inertie lié à la courbure moyenne de la surface moyenne déformée – et un problème membranaire totalement ou partiellement couplé.

En ce qui concerne le traitement numérique des modèles de plaque déduits, nous choisissons de concentrer notre attention sur le problème de flexion, dont la structure est la même dans tous les quatre modèles et dont l'étude mathématique et numérique, compte tenu de sa formulation, est d'intérêt en tant que tel. Pour cela, après avoir donné la preuve d'existence et unicité de la solution en s'inspirant de l'approche de Raviart et Thomas [54], nous effectuons une étude numérique du problème en utilisant une discrétisation conforme en espace avec des *éléments finis de classe* C^1 – en particulier, nous utilisons des éléments de type HCT (Hsieh-Clough-Tocher), voir e.g. [15] ; la discrétisation en temps est de type *Newmark*. Nous remarquons que l'utilisation d'éléments HCT requiert l'application d'un schéma de quadrature opportun dans l'implémentation de la discrétisation. Notre analyse numérique est ensuite validée avec des tests numériques effectués sous l'environnement `FreeFEM++` [37].

Dans le dernier chapitre, nous présentons succinctement des perspectives futures de recherche pour ce qui concerne les aspects numériques. En effet, il est connu qu'un traitement numérique avec des éléments finis de classe C^1 est cher du point de vue computationnel. Il est alors intéressant d'utiliser une méthode non conforme (mixte ou hybride) ; nous nous concentrons sur une méthode de type HHO (voir e.g. [18], [19]) et décrivons une première approche du problème de flexion rencontré dans les chapitres précédents avec cette méthode.

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Notations et Conventions

Tout au long de ce manuscrit, une fonction et sa valeur seront notées avec la même lettre. On notera les quantités scalaires et les points en caractères ordinaires, les champs vectoriels et tensoriels (quel que soit leur ordre) en gras. On conviendra d'utiliser le mot *tenseur* comme synonyme de "transformation linéaire entre espaces vectoriels", comme d'habitude en mécanique des milieux continus.

Soit $\mathbb{E}^3$ un espace affine euclidien tridimensionnel et $\mathbb{V}^3$ son espace vectoriel des translations associé. Dans toute la thèse on supposera qu'un repère cartesien $\mathcal{R} = \{o; \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$, avec $o \in \mathbb{E}^3$ et $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ une base orthonormée de $\mathbb{V}^3$, ait été fixé une fois pour toutes dans $\mathbb{E}^3$. Pour simplicité de notation, $\mathbb{R}^3$ désignera alors l'un des ensembles $\mathbb{E}^3$, $\mathbb{V}^3$ ou $\mathbb{R}^3$ même, selon le contexte (on écrira $x \in \mathbb{R}^3$ pour un point et $\mathbf{v} \in \mathbb{R}^3$ pour un vecteur). De même, on ne fera pas de distinction entre un tenseur et sa représentation en composantes (matricielle pour les tenseurs du second ordre) par rapport à la base cartesienne fixée. Les indices latins prennent leurs valeurs dans l'ensemble $\{1, 2, 3\}$, les indices grecs dans l'ensemble $\{1, 2\}$. On utilise parfois la convention d'Einstein sur la sommation sur les indices répétés.

Ci-après nous donnons la liste des notations principales utilisées, sauf avis contraire explicite.

— $\mathbb{N}^* := \mathbb{N} \backslash \{0\}$, $\mathbb{R}^* := \mathbb{R} \backslash \{0\}$.

— $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geqslant 0\}$, $\mathbb{R}_+^* := \mathbb{R}_+ \backslash \{0\}$.

— Ω : sous-ensemble ouvert, borné, connexe de $\mathbb{R}^3$, avec frontière $\partial\Omega$ Lipschitz-continue.

— $\overline{\Omega} := \Omega \cup \partial\Omega$: fermeture de Ω.

— (x_1, x_2, x_3) : coordonnées cartesiennes d'un point $x \in \overline{\Omega}$.

— $v_{|U}$: restriction à $U \subseteq \overline{\Omega}$ de la fonction v définie sur $\overline{\Omega}$ à valeurs réelles.

— $\partial^{\boldsymbol{\alpha}} v := \dfrac{\partial^{|\boldsymbol{\alpha}|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}$: dérivée partielle d'ordre $m \geqslant 1$ de v, où $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ est un multi-indice satisfaisant $|\boldsymbol{\alpha}| := \alpha_1 + \alpha_2 + \alpha_3 = m$.

— $\partial_i v := \partial v / \partial x_i$, $\partial_{ij} v := \partial^2 v / \partial x_i \partial x_j$: dérivées partielles de premier et second ordre de v.

— ∇v : gradient de $v : \Omega \to \mathbb{R}$.

— $\dot{v}$, $\partial_t v$: dérivée temporelle [1] de v définie sur $\Omega \times (0, T)$, $T > 0$.

1. Puisque toutes les formulations sont valides dans un domaine de référence fixe, il s'agit toujours d'une dérivée partielle.

— $\mathbf{u} = (u_i)$: représentation en composantes d'un vecteur.

— $\mathbb{S}^2 \coloneqq \{\mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}| = 1\}$.

— $\mathbf{u} \cdot \mathbf{v}$, $|\mathbf{u}|$, $\mathbf{u} \times \mathbf{v}$, $\mathbf{u} \otimes \mathbf{v}$: produit scalaire euclidien de $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, norme euclidienne de $\mathbf{u}$, produit vectoriel de $\mathbf{u}$ et $\mathbf{v}$, produit tensoriel de $\mathbf{u}$ et $\mathbf{v}$.

— $\nabla \mathbf{u}$, $\operatorname{div} \mathbf{u}$, $\nabla \times \mathbf{u}$: gradient, divergence et rotationnel de $\mathbf{u} \colon \Omega \to \mathbb{R}^3$.

— $\operatorname{End}(E)$: espace des endomorphismes d'un espace normé E.

— Lin : espace vectoriel des tenseurs du second ordre (endomorphismes de $\mathbb{R}^3$).

— **Lin** : espace vectoriel des tenseurs du troisième ordre (applications linéaires de $\mathbb{R}^3$ dans Lin).

— $\mathbb{L}\mathrm{in}$: espace vectoriel des tenseurs du quatrième ordre (endomorphismes de Lin).

— $\mathbf{A} = (A_{ij})$: représentation en composantes d'un tenseur $\mathbf{A} \in \mathrm{Lin}$.

— $\mathbf{B} = (B_{ijk})$: représentation en composantes d'un tenseur $\mathbf{B} \in \mathbf{Lin}$.

— $\mathbf{C} = (C_{ijk\ell})$: représentation en composantes d'un tenseur $\mathbf{C} \in \mathbb{L}\mathrm{in}$.

— $\operatorname{tr} \mathbf{A}$, $\det \mathbf{A}$, $\mathbf{A}^*$, $\mathbf{A}^T$: trace, déterminant, cofacteur et transposé d'un tenseur $\mathbf{A} \in \mathrm{Lin}$.

— $\mathbf{A} : \mathbf{B} \coloneqq \operatorname{tr}(\mathbf{A}\mathbf{B}^T) = \sum_{i,j=1}^{3} A_{ij} B_{ij}$: produit scalaire tensoriel de $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$.

— $\|\mathbf{A}\|_2 \coloneqq \sqrt{\mathbf{A} : \mathbf{A}}$: norme d'un tenseur $\mathbf{A} \in \mathrm{Lin}$ induite par le produit scalaire tensoriel.

— Sym : sous-espace de Lin des tenseurs symétriques.

— Skw : sous-espace de Lin des tenseurs anti-symétriques.

— Sym^+ : ensemble des tenseurs symétriques et définis positifs.

— $\operatorname{sym} \mathbf{A}$, $\operatorname{skw} \mathbf{A}$: parties symétrique et anti-symétrique de $\mathbf{A} \in \mathrm{Lin}$.

— $\operatorname{div} \mathbf{A}$: divergence du champ tensoriel $\mathbf{A} \colon \Omega \to \mathrm{Lin}$.

— $\mathbf{e} = \mathbf{e}(\mathbf{u}) = \operatorname{sym} \nabla \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$: tenseur de déformation linéarisé.

— $C^\infty(\overline{\Omega})$: espace des fonctions indéfiniment différentiables sur $\overline{\Omega}$.

— $C^m([0,T]; H)$: espace des fonctions différentiables avec différentielle continue jusqu'à l'ordre m, de l'intervalle réel $[0,T]$ à valeurs dans un espace de Hilbert H.

— $\|v\|_{C^m([0,T];H)} \coloneqq \max_{0 \leqslant k \leqslant m} \left(\sup_{0 \leqslant t \leqslant T} \left\| \frac{\mathrm{d}^k v}{\mathrm{d}t^k}(t) \right\|_H \right)$.

— $L^p(U)$: espace de Lebesgue des fonctions scalaires, avec $1 \leqslant p \leqslant \infty$ et $U \subseteq \overline{\Omega}$.

— $\mathbf{L}^p(U) \coloneqq [L^p(U)]^k$: espace de Lebesgue des fonctions vectorielles ou tensorielles, avec $k \geqslant 3$ entier opportun, selon le contexte.

— $\|v\|_{L^p(\Omega)} \coloneqq \left(\int_\Omega |v(x)|^p \, \mathrm{d}x \right)^{1/p}$ si $p < +\infty$;
$\|v\|_{L^\infty(\Omega)} \coloneqq \inf\{C \geqslant 0 : |v(x)| \leqslant C \text{ pour presque tout } x \in \Omega\}$.

— $\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)} \coloneqq \left(\sum_{i=1}^{3}\int_\Omega |v_i(x)|^p\,\mathrm{d}x\right)^{1/p}$ si $p < +\infty$;
$\|\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)} \coloneqq \inf\{C \geqslant 0 : |\mathbf{v}(x)| \leqslant C$ pour presque tout $x \in \Omega\}$.

— $\|\mathbf{A}\|_{\mathbf{L}^p(\Omega)} \coloneqq \left(\sum_{i,j=1}^{3}\int_\Omega |A_{ij}(x)|^p\,\mathrm{d}x\right)^{1/p}$ si $p < +\infty$;
$\|\mathbf{A}\|_{\mathbf{L}^\infty(\Omega)} \coloneqq \inf\{C \geqslant 0 : \|\mathbf{A}(x)\|_2 \leqslant C$ pour presque tout $x \in \Omega\}$, où $\mathbf{A}\colon \Omega \to$ Lin. On peut évidemment remplacer $\|\cdot\|_2$ par une norme matricielle quelconque.

— $W^{m,p}(\Omega)$: espace de Sobolev des fonctions scalaires, $1 \leqslant p \leqslant \infty$, $m \geqslant 1$ entier.

— $\mathbf{W}^{m,p}(\Omega) \coloneqq [W^{m,p}(\Omega)]^3$: espace de Sobolev des fonctions vectorielles, $1 \leqslant p \leqslant \infty$, $m \geqslant 1$ entier.

— $\|v\|_{W^{m,p}(\Omega)} \coloneqq \left(\int_\Omega (|v(x)|^p + \sum_{|\alpha|\leqslant m} |\partial^\alpha v(x)|^p)\,\mathrm{d}x\right)^{1/p}$ si $p < +\infty$;
$\|v\|_{W^{m,\infty}(\Omega)} \coloneqq \max\{\|v\|_{L^\infty(\Omega)}, \|\nabla v\|_{\mathbf{L}^\infty(\Omega)}\}$.

— $\|\mathbf{v}\|_{\mathbf{W}^{m,p}(\Omega)} \coloneqq \left(\sum_{i=1}^{3}\int_\Omega (|v_i(x)|^p + \sum_{|\alpha|\leqslant m} |\partial^\alpha v_i(x)|^p)\,\mathrm{d}x\right)^{1/p}$ si $p < +\infty$;
$\|\mathbf{v}\|_{\mathbf{W}^{m,\infty}(\Omega)} \coloneqq \max\{\|\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)}, \|\nabla\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)}\}$.

— $H^m(\Omega) \coloneqq W^{m,2}(\Omega)$, $\mathbf{H}^m(\Omega) \coloneqq \mathbf{W}^{m,2}(\Omega)$.

— $\mathbf{H}(\mathrm{curl}, \Omega) \coloneqq \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \nabla \times \mathbf{v} \in \mathbf{L}^2(\Omega)\}$, où $\nabla \times \mathbf{v}$ est pris au sens des distributions.

— $\|\mathbf{v}\|_{\mathbf{H}(\mathrm{curl},\Omega)} \coloneqq \left(\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \times \mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}\right)^{1/2}$.

— $|v|_{0,\Omega} \coloneqq \|v\|_{L^2(\Omega)}$, $\|v\|_{m,\Omega} \coloneqq \|v\|_{H^m(\Omega)}$.

— $|\mathbf{v}|_{0,\Omega} \coloneqq \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}$, $\|\mathbf{v}\|_{m,\Omega} \coloneqq \|\mathbf{v}\|_{\mathbf{H}^m(\Omega)}$.

— $\mathcal{D}(\Omega)$, $\mathcal{D}(0, T)$: espaces des fonctions indéfiniment différentiables à support compact inclus, respectivement, dans Ω et dans $(0, T)$.

— $W_0^{m,p}(\Omega)$: fermeture de $\mathcal{D}(\Omega)$ dans $W^{m,p}(\Omega)$.

— $\mathbf{W}_0^{m,p}(\Omega)$: fermeture de $[\mathcal{D}(\Omega)]^3$ dans $\mathbf{W}^{m,p}(\Omega)$.

— $H_0^m(\Omega)$: fermeture de $\mathcal{D}(\Omega)$ dans $H^m(\Omega)$.

— $\mathbf{H}_0^m(\Omega)$: fermeture de $[\mathcal{D}(\Omega)]^3$ dans $\mathbf{H}^m(\Omega)$.

— $L^p(0, T; H) \coloneqq \left\{v\colon (0,T) \to H,\ \int_0^T \|v(t)\|_H^p\,\mathrm{d}t < +\infty\right\}$,
où $1 \leqslant p < \infty$ et H est un espace de Hilbert sur $\mathbb{R}$ muni de la norme $\|\cdot\|_H$;
$L^\infty(0, T; H) \coloneqq \{v\colon (0,T) \to H : \exists C \geqslant 0$ tel que
$\|v(t)\|_H \leqslant C$ pour presque tout $t \in (0, T)\}$.

— $\|v\|_{L^p(0,T;H)} \coloneqq \left(\int_0^T \|v(t)\|_H^p\,\mathrm{d}t\right)^{1/p}$ si $p < +\infty$;
$\|v\|_{L^\infty(0,T;H)} \coloneqq \inf\{C \geqslant 0 : \|v(t)\|_H \leqslant C$ pour presque tout $t \in (0, T)\}$.

— $H^1(0,T;H) \coloneqq \{v \in L^2(0,T;H) : v' \in L^2(0,T;H)\}$, v' désignant la dérivée faible de v.

— $\|v\|_{H^1(0,T;H)} \coloneqq \left(\|v\|^2_{L^2(0,T;H)} + \|v'\|^2_{L^2(0,T;H)}\right)^{1/2}$.

— $H^2(0,T;H) \coloneqq \{v \in H^1(0,T;H) : v' \in H^1(0,T;H)\}$.

— $\|v\|_{H^2(0,T;H)} \coloneqq \left(\|v\|^2_{H^1(0,T;H)} + \|v''\|^2_{L^2(0,T;H)}\right)^{1/2}$, où v'' désigne la dérivée faible de v'.

— H^* : espace dual de l'espace de Hilbert H.

— $\langle \cdot, \cdot \rangle_{H^*,H}$: crochet de dualité entre H^* et H.

— $\mathcal{L}(H_1, H_2)$: espace des opérateurs linéaires et continus de H_1 dans H_2, avec H_1 et H_2 des espaces de Hilbert. On écrit $\mathcal{L}(H)$ lorsque $H_1 = H_2 = H$.

— $\|A\|_{\mathcal{L}(H_1,H_2)} \coloneqq \sup_{u \in H_1 \setminus \{0\}} \dfrac{\|Au\|_{H_2}}{\|u\|_{H_1}}$.

— $R(A)$: image de l'opérateur linéaire $A \colon H_1 \to H_2$.

— $\rho(A) \coloneqq \{\lambda \in \mathbb{R} : A - \lambda I$ est une bijection de H sur $H\}$: ensemble résolvant de l'opérateur linéaire $A \colon H \to H$.

— $u_n \rightharpoonup u$: convergence faible de la suite $\{u_n\}_{n\in\mathbb{N}} \subset H$ vers $u \in H$.

Première partie

Préliminaires

Chapitre 1

Éléments d'Analyse Fonctionnelle

Nous rappelons dans ce premier chapitre les outils mathématiques essentiels utilisés dans la suite de ce document, sans en présenter un exposé détaillé, pour lequel nous renvoyons aux ouvrages cités. Le chapitre est divisé en trois sections. La première est consacrée au rappel de certaines propriétés générales des espaces de Sobolev : théorème de compacité, théorème de trace, inégalités de Poincaré-Friedrichs et de Korn ; en particulier, le théorème de compacité est utilisé dans le Chapitre 5, le théorème de trace et les inégalités de Poincaré-Friedrichs et de Korn dans les Chapitres 3 et 4. Dans la deuxième section, nous rappelons quelques résultats portant sur les opérateurs compacts autoadjoints et définis positifs, utilisés dans le Chapitre 5. Enfin, des éléments de théorie des semi-groupes et le théorème de Hille-Yosida, utilisé dans le Chapitre 3, font l'objet de la troisième section.

Une présentation détaillée pour ce qui concerne les espaces de Sobolev peut se trouver dans les ouvrages de Brezis [11], Girault et Raviart [31] ou Nečas [47]. Des chapitres consacrés, en général, à la théorie des opérateurs compacts autoadjoints se trouvent, e.g., dans les ouvrages de Brezis [11], Raviart et Thomas [54], Sanchez Hubert et Sanchez Palencia [56]. Ce dernier ouvrage, avec ceux de Brezis [11] et de Pazy [48], donne aussi une présentation de la théorie des semi-groupes et de son application à la résolution d'équations différentielles abstraites (portant sur des inconnues à valeurs dans un espace de Hilbert).

1.1 Espaces de Sobolev

On rappelle dans cette section quelques propriétés générales des espaces de Sobolev.

Théorème 1.1. *Soit $m \in \mathbb{N}^*$. Pour tout $1 \leqslant p < +\infty$, $W^{m,p}(\Omega)$ est un espace de Banach. Lorsque $p = 2$, $W^{m,2}(\Omega) = H^m(\Omega)$ est un espace de Hilbert. Le même résultat vaut respectivement pour $\mathbf{W}^{m,p}(\Omega)$ et pour $\mathbf{H}^m(\Omega)$.*

Proposition 1.1. *Soit $m \in \mathbb{N}^*$. Pour tout $1 < p < +\infty$, $W^{m,p}(\Omega)$ est un espace réflexif. Le même vaut pour $\mathbf{W}^{m,p}(\Omega)$.*

Théorème 1.2 (Rellich-Kondrakov)**.** *Pour tout $1 \leqslant p \leqslant +\infty$, l'injection canonique $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ est compacte ; autrement dit, toute partie bornée de $W^{1,p}(\Omega)$ est relativement compacte dans $L^p(\Omega)$. Le même résultat vaut pour $\mathbf{W}^{1,p}(\Omega)$ et $\mathbf{L}^p(\Omega)$.*

Rappelons en outre le théorème suivant, qui permet de donner un sens aux valeurs au bord $\partial\Omega$ pour des fonctions $v \in H^m(\Omega)$ à travers la notion de trace.

Théorème 1.3. *Soit $m \in \mathbb{N}^*$. Il existe un unique opérateur linéaire et continu $T\colon H^m(\Omega) \to L^2(\partial\Omega)$ tel que, pour tout $x \in \partial\Omega$ et pour toute fonction $v \in C^\infty(\overline{\Omega})$,*

$$(Tv)(x) = v(x).$$

On appelle T opérateur de trace.

Corollaire 1.1. *Soit α un multi-indice tel que $|\alpha| \leqslant m-1$. Alors il existe un unique opérateur linéaire et continu $T_\alpha\colon H^m(\Omega) \to L^2(\partial\Omega)$ tel que pour toute fonction $v \in C^\infty(\overline{\Omega})$,*

$$T_\alpha v = \partial^\alpha v.$$

Grâce à la notion de trace, on peut alors donner un sens aux valeurs au bord $\partial\Omega$ des fonctions de l'espace $H^m(\Omega)$, ainsi que de toutes leurs dérivées jusqu'à l'ordre $m-1$. L'espace $H_0^1(\Omega)$ peut donc être caractérisé comme le noyau de l'opérateur de trace $T\colon H^1(\Omega) \to L^2(\partial\Omega)$, c'est à dire

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v_{|\partial\Omega} = 0\}, \quad \text{où} \quad v_{|\partial\Omega} := Tv.$$

De façon analogue, en écrivant v au lieu de Tv et de même pour les dérivées $\partial_i v$, et en notant $\partial_n v := \sum_{i=1}^3 n_i \partial_i v$ la dérivée normale de v sur $\partial\Omega$, on a

$$H_0^2(\Omega) = \{v \in H^2(\Omega) : v = \partial_n v = 0 \text{ sur } \partial\Omega\}.$$

Rappelons ensuite les inégalités de Poincaré-Friedrichs et de Korn, deux outils essentiels dans l'approche variationnelle des problèmes aux limites. Considérons la semi-norme

$$H_0^m(\Omega) \ni v \mapsto |v|_{m,\Omega} := \left(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v(x)|^2 \,\mathrm{d}x \right)^{1/2},$$

où v peut être considérée comme fonction scalaire ou vectorielle. On a alors le résultat suivant.

Théorème 1.4 (Inégalité de Poincaré-Friedrichs)**.** *Pour tout $m \in \mathbb{N}^*$, il existe une constante $C(m,\Omega) > 0$ telle que*

$$\|v\|_{m,\Omega} \leqslant C(m,\Omega)|v|_{m,\Omega}, \quad \forall v \in H_0^m(\Omega).$$

On en déduit l'équivalence de la semi-norme $|\cdot|_{m,\Omega}$ à la norme $\|\cdot\|_{m,\Omega}$ sur l'espace $H_0^m(\Omega)$.

Étant donnée $\mathbf{v}\colon \Omega \to \mathbb{R}^3$, l'inégalité de Korn fait intervenir le gradient symétrique $\mathbf{e}(\mathbf{v}) := \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)$. Soit $\Gamma_0 \subset \partial\Omega$ une partie de la frontière de Ω de mesure strictement positive. En notant $\mathbf{H}^1(\Omega,\Gamma_0) := \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} = \mathbf{0} \text{ sur } \Gamma_0\}$, le résultat s'énonce de la façon suivante.

Théorème 1.5 (Inégalité de Korn)**.** *Il existe une constante $C(\Omega,\Gamma_0) > 0$ telle que*

$$\|\mathbf{v}\|_{1,\Omega} \leqslant C(\Omega,\Gamma_0)|\mathbf{e}(\mathbf{v})|_{0,\Omega}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega,\Gamma_0).$$

On a alors que la semi-norme

$$\mathbf{H}^1(\Omega,\Gamma_0) \ni \mathbf{v} \mapsto |\mathbf{e}(\mathbf{v})|_{0,\Omega}$$

est une norme équivalente à la norme $\|\cdot\|_{1,\Omega}$ sur $\mathbf{H}^1(\Omega,\Gamma_0)$. Une démonstration de l'inégalité de Korn est donnée, e.g., dans l'ouvrage de Duvaut et Lions [23].

1.2 Opérateurs Compacts, Autoadjoints et Positifs

Considérons un espace de Hilbert H sur les réels, séparable, muni du produit scalaire $(\cdot,\cdot)_H$ et de la norme associée $\|\cdot\|_H$, une suite $\{u_n\}_{n\in\mathbb{N}} \subset H$ d'éléments de H et un opérateur linéaire $\Lambda\colon u \in H \mapsto \Lambda u \in H$ satisfaisant les hypothèses suivantes :

(a) Λ est *compact*, i.e.

$$u_n \rightharpoonup u \quad \text{dans } H \quad \Longrightarrow \quad \Lambda u_n \to \Lambda u \quad \text{dans } H;$$

(b) Λ est *autoadjoint*, i.e.

$$(\Lambda u, v)_H = (u, \Lambda v)_H, \quad \forall u, v \in H;$$

(c) Λ est *défini positif*, i.e.

$$(\Lambda u, u)_H \geqslant 0 \ \ \forall u \in H \quad \text{et} \quad \Lambda u \neq 0 \text{ si } u \neq 0.$$

En particulier, la compacité de Λ implique que Λ est continu, soit $\Lambda \in \mathcal{L}(H)$.

On rappelle que les valeurs propres d'un opérateur autoadjoint sont réelles et que deux vecteurs propres associés à deux valeurs propres différentes sont orthogonaux. L'hypothèse (c) implique, en particulier, que toutes les valeurs propres de Λ sont strictement positives.

Théorème 1.6. *Sous les hypothèses* (a), (b), (c) *il existe une suite décroissante de valeurs propres de Λ tendant vers zéro :*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots \to 0.$$

Pour chaque valeur propre λ_i, le sous-espace propre (fermé)

$$E_i \coloneqq \{u \in H : \Lambda u = \lambda_i u\}$$

est de dimension finie (λ_i est de multiplicité finie), et les sous-espaces E_i sont mutuellement orthogonaux. En convenant que chaque λ_i soit répété autant de fois que sa multiplicité, on obtient une base orthonormale $\{e_k\}_{k\in\mathbb{N}}$ de H formée de vecteur propres de Λ.

Introduisons maintenant un autre espace de Hilbert séparable V, avec produit scalaire $(\cdot,\cdot)_V$ et norme associée $\|\cdot\|_V$, tel que $V \subset H$ avec injection compacte et dense (c'est à dire, dont l'image est dense dans H). Soient V^* et H^* les espaces duaux de V et de H, respectivement. En identifiant H à son dual, grâce au théorème de Riesz-Fréchet, on a :

$$V \subset H \simeq H^* \subset V^*,$$

où l'injection $H^* \subset V^*$ est compacte et dense. On a clairement

$$\langle f, v\rangle_{V^*,V} = (f, v)_H \quad \text{lorsque } f \in H.$$

Considérons le produit scalaire $(\cdot,\cdot)_V$ comme une forme bilinéaire $a\colon V \times V \to \mathbb{R}$ symétrique, continue et coercive (i.e. il existe $M, \alpha > 0$ tels que $|a(u,v)| \leqslant M\|u\|_V\|v\|_V$, $|a(v,v)| \geqslant \alpha\|v\|_V^2$ $\forall u, v \in V$) :

$$a(u, v) = (u, v)_V, \quad \forall u, v \in V.$$

Réciproquement, étant donnée une forme bilinéaire dans V possédant de telles propriétés, on peut la considérer comme produit scalaire dans V. D'après le théorème de Riesz-Fréchet, pour tout $u \in V$ il existe alors un unique $f \in V^*$ tel que

$$a(u, v) = (u, v)_V = \langle f, v \rangle_{V^*,V}, \quad \forall v \in V.$$

Cela permet de définir les opérateurs $A \in \mathcal{L}(V, V^*)$ et $A^{-1} \in \mathcal{L}(V^*, V)$ tels que

$$Au = f, \quad u = A^{-1} f,$$

et d'écrire alors

$$a(u, v) = \langle Au, v \rangle_{V^*,V} = \langle f, v \rangle_{V^*,V}.$$

Comme $a(\cdot, \cdot)$ est le produit scalaire sur V, l'opérateur A définit une isométrie entre V et V^*, et on a :

$$\|A\|_{\mathcal{L}(V,V^*)} = \sup_{u \in V \setminus \{0\}} \frac{\|Au\|_{V^*}}{\|u\|_V} < M, \quad \|A^{-1}\|_{\mathcal{L}(V^*,V)} = \sup_{f \in V^* \setminus \{0\}} \frac{\|A^{-1} f\|_V}{\|f\|_{V^*}} < \frac{1}{\alpha}.$$

On définit maintenant la *restriction A_H de A à H* comme la restriction de A au domaine

$$D(A_H) := \{v \in V \subset H : Av \in H\};$$

c'est à dire, $D(A_H)$ contient les éléments de V, regardés comme éléments de H, tels que leurs images soient des éléments de $H \subset V^*$. On a donc

$$A_H v = Av, \quad \forall v \in D(A_H).$$

L'opérateur inverse $A_H^{-1} \colon H \to D(A_H) \subset H$ est alors bien défini, et on peut montrer que A_H^{-1} possède les propriétés (a), (b) et (c). En appliquant donc le Théorème 1.6 à A_H, on déduit l'existence d'une infinité dénombrable de valeurs propres strictement positives de A_H, notées $1/\lambda_i = 1/\omega_i^2$, telles que $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ et que les vecteurs propres associés constituent une base orthonormale de H. Grâce à ce résultat, on peut finalement démontrer le théorème suivant.

Théorème 1.7. *L'opérateur A_H possède une infinité dénombrable de valeurs propres strictement positives*

$$\lambda_i = \omega_i^2, \quad i = 1, 2, \ldots,$$

telles que

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots \to +\infty,$$

où chaque valeur propre est répétée autant de fois que sa multiplicité. Les vecteurs propres correspondants e_i peuvent être choisis de telle sorte qu'ils forment une base des espaces H, V et V^* orthonormale dans H et orthogonale dans V et V^*, avec

$$\|e_i\|_H^2 = 1, \quad \|e_i\|_V^2 = \omega_i^2, \quad \|e_i\|_{V^*}^2 = \frac{1}{\omega_i^2}.$$

Ainsi, étant donné un élément $v \in V$, on peut écrire

$$v = \sum_{k \in \mathbb{N}} V_k e_k, \quad \text{avec} \quad \|v\|_V^2 = \sum_{k \in \mathbb{N}} V_k^2 \omega_k^2 < +\infty,$$

tandis que, pour $f \in V^*$, on a

$$f = \sum_{k \in \mathbb{N}} F_k e_k \quad \text{avec} \quad \|f\|_{V^*}^2 = \sum_{k \in \mathbb{N}} \frac{F_k^2}{\omega_k^2} < +\infty.$$

1.3 Théorie des Semi-groupes

La notion de semi-groupe généralise la formule usuelle utilisée pour résoudre des problèmes d'évolutions de la forme

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = 0, \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

où $u\colon \mathbb{R}_+ \to E$ est une fonction à valeurs dans un espace normé E de norme $\|\cdot\|_E$ et de dimension finie. Si $A \in \mathrm{End}(E)$, on sait que la solution de ce système s'écrit sous la forme

$$u(t) = e^{-At}u_0,$$

$\mathbb{R}_+ \ni t \mapsto e^{-At} \in \mathrm{End}(E)$ étant l'exponentielle d'endomorphisme. On veut retrouver une formule analogue lorsque E est remplacé par un espace de Hilbert H de dimension infinie. On donne d'abord la définition suivante.

Définition 1.1. Soit H un espace de Hilbert de norme $\|\cdot\|_H$. On appelle *semi-groupe continu de contractions sur H* (ou simplement *semi-groupe*) une famille d'opérateurs $\{S(t)\}_{t\geqslant 0} \subset \mathcal{L}(H)$ possédant les propriétés suivantes :

(a) $\|S(t)\|_{\mathcal{L}(H)} \leqslant 1 \quad \forall t \geqslant 0$ (contraction) ;

(b) $S(0) = I, \quad S(t_1 + t_2) = S(t_1) \circ S(t_2) \quad \forall t_1, t_2 \geqslant 0$;

(c) $\lim_{t\to 0^+} \|S(t)u - u\|_H = 0 \quad \forall u \in H$ (continuité en 0).

La propriété (a) est liée au fait que, souvent, l'énergie de la solution de (1.1) (dont $t \mapsto \|u(t)\|_E^2$ fournit une mesure) décroît au cours de l'évolution : $\|u(t_1)\|_E \leqslant \|u(t_2)\|_E$ pour $t_2 > t_1$. De plus, on déduit de (b) et de (c) que la fonction $t \mapsto S(t)u \in H$ est en fait continue sur $\mathbb{R}_+$, pour tout $u \in H$. Rappelons maintenant la notion de générateur infinitésimal d'un semi-groupe.

Définition 1.2. Le *générateur infinitésimal* $-A$ du semi-groupe $\{S(t)\}_{t\geqslant 0}$ est l'opérateur défini par

$$-Au = \lim_{t\to 0^+} \frac{S(t)u - u}{t} \quad \text{fortement dans } H;$$

le *domaine* de A est alors défini comme l'ensemble

$$D(A) = \left\{ u \in H : \exists \lim_{t\to 0^+} \frac{S(t)u - u}{t} \in H \right\}.$$

On a les trois résultats suivants.

Lemme 1.1. *Pour tout $u \in D(A)$, la dérivée de la fonction $\mathbb{R}_+^* \ni t \mapsto S(t)u \in H$ existe dans le sens de la norme $\|\cdot\|_H$ et on a*

$$\frac{\mathrm{d}S(t)u}{\mathrm{d}t} = -S(t)Au = -AS(t)u, \quad t > 0; \tag{1.2}$$

de plus,

$$S(t)u - u = -\int_0^t S(\tau)Au\,\mathrm{d}\tau.$$

Lemme 1.2. *Pour tout $u \in H$ et $t \in \mathbb{R}_+$, on a*

$$S(t)u - u = -A \int_0^t S(\tau)u \,\mathrm{d}\tau.$$

Lemme 1.3. *L'opérateur A est fermé et $D(A)$ est dense dans H.*

Ainsi, la fonction $u\colon \mathbb{R}_+ \ni t \mapsto u(t) = S(t)u_0 \in H$ est la solution du problème aux valeurs initiales (1.1) lorsque $-A$ est le générateur du semi-groupe $\{S(t)\}_{t\geqslant 0}$ et que $u_0 \in D(A)$. Le lemme suivant donne une condition nécessaire pour que $-A$ soit le générateur d'un semi-groupe de contractions.

Lemme 1.4. *Si $-A$ est le générateur d'un semi-groupe de contractions, alors A est* accrétif*, c'est à dire*

$$(Au, u)_H \geqslant 0, \quad \forall u \in D(A).$$

Le théorème suivant donne une caractérisation des générateurs de semi-groupes de contractions.

Théorème 1.8 (Lumer-Phillips)**.**

(i) Si A est un opérateur accrétif et qu'il existe $\lambda > 0$ tel que $R(A + \lambda I) = H$ (i.e., $A + \lambda I$ est surjectif), alors $-A$ est le générateur d'un unique semi-groupe de contractions.

(ii) Si $-A$ est le générateur d'un semi-groupe de contractions, alors A est accrétif et $R(A + \lambda I) = H$ pour tout $\lambda > 0$. De plus, $-\lambda \in \rho(A)$.

Les deux propositions suivantes sont des conséquences du théorème de Lumer-Phillips.

Proposition 1.2.

(i) Soit A un opérateur surjectif de $D(A)$ dans H, tel que $(Au, u)_H \geqslant C\|u\|_H^2$ pour tout $u \in D(A)$, pour une constante $C > 0$. Alors $-A$ est le générateur d'un semi-groupe de contractions.

(ii) Le même résultat vaut pour un opérateur A accrétif tel que $0 \in \rho(A)$.

Proposition 1.3. *Soit A un opérateur satisfaisant les deux conditions suivantes :*

(i) il existe une constante $C \geqslant 0$ telle que

$$(Au, u)_H + C\|u\|_H^2 \geqslant 0, \quad \forall u \in D(A);$$

(ii) il existe une constante $\beta > C$ telle que $R(A + \beta I) = H$.

Alors $-A$ est le générateur d'un semi-groupe fortement continu (non nécessairement de contractions).

On définit maintenant la notion d'opérateur maximal monotone.

Définition 1.3. Soit $A\colon D(A) \subset H \to H$ un opérateur linéaire (non nécessairement continu). On dit que A est *maximal monotone* si A est accrétif et $I + A$ est surjectif, c'est à dire, respectivement :

$$(Au, u)_H \geqslant 0 \ \ \forall u \in D(A), \quad R(I + A) = H.$$

Grâce au théorème de Lumer-Phillips, étant donné un semi-groupe continu de contractions $\{S(t)\}_{t\geqslant 0}$, il existe un unique opérateur maximal monotone A tel que $-A$ soit le générateur infinitésimal de $\{S(t)\}_{t\geqslant 0}$, et on écrit alors $S(t) = S_A(t)$.

On conclut cette section en donnant l'énoncé du théorème de Hille-Yosida, ainsi qu'une version de ce théorème adaptée aux problèmes d'évolutions non-homogènes.

Théorème 1.9 (Hille-Yosida)**.** *Soit A un opérateur maximal monotone dans un espace de Hilbert H. Alors, pour tout $u_0 \in D(A)$, il existe une unique fonction*

$$u \in C^1(\mathbb{R}^+; H) \cap C^0(\mathbb{R}^+; D(A))$$

vérifiant (1.1). *De plus, on a les estimations*

$$\|u(t)\|_H \leqslant \|u_0\|_H, \quad \left\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right\|_H = \|Au(t)\|_H \leqslant \|Au_0\|_H, \quad \forall t \geqslant 0.$$

Théorème 1.10. *Soit H un espace de Hilbert, $A\colon D(A) \subset H \to H$ un opérateur linéaire maximal monotone, et $T > 0$. Alors, pour tout $u_0 \in D(A)$ et pour tout $f \in C^1([0,T]; H)$, le problème*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) + Au(t) = f(t), & t > 0, \\ u(0) = u_0 \end{cases}$$

admet une unique solution

$$u \in C^1([0,T]; H) \cap C^0([0,T]; D(A)).$$

De plus, u est donnée par la formule

$$u(t) = S_A(t)u_0 + \int_0^t S_A(t-s)f(s)\,\mathrm{d}s,$$

$\{S_A(t)\}_{t\geqslant 0}$ étant le semi-groupe de contractions engendré par $-A$.

Chapitre 2

Éléments de Mécanique des Milieux Continus

Dans ce chapitre on rappelle quelques notions de mécanique des milieux continus utilisées dans le Chapitre 3. La première section est consacrée à des rappels de thermodynamique classique (bilan de l'énergie, inégalité de Clausius-Duhem, énergie libre de Helmholtz, inégalité de dissipation réduite, lois de comportement admissibles). Concernant ces notions, on mentionne comme références les ouvrages de Frémond [25] et de Gurtin, Fried et Anand [34], et l'article de Coleman et Noll [16]. Dans la deuxième section on rappelle la notion de tenseur acoustique et son rôle dans l'étude de la propagation d'ondes dans un milieu élastique et linéaire. Un exposé détaillé de ces concepts peut se trouver dans l'article de Gurtin [33].

2.1 Thermodynamique Classique

Considérons un corps continu occupant la région d'espace $\Omega \subset \mathbb{R}^3$ dans sa configuration de référence ; soit $\mathcal{P} \subset \Omega$ une partie arbitraire de Ω.

Soit $h\colon \Omega \times \mathbb{R}_+^* \to \mathbb{R}$ une source de chaleur volumique et $\mathbf{q}\colon \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}^3$ un flux de chaleur. On suppose d'abord que le milieux est *rigide* et en état d'équilibre mécanique (c'est à dire, il n'est sollicité par aucun système de chargements). Le bilan de l'énergie sous forme intégrale s'écrit alors

$$\dot{E}(\mathcal{P}) = Q(P) \quad \text{pour toute partie } \mathcal{P} \subset \Omega,$$

où $E(\mathcal{P})$ est l'*énergie interne* de la partie $\mathcal{P}$, et

$$Q(\mathcal{P}) := -\int_{\partial\mathcal{P}} \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}\mathcal{S} + \int_{\mathcal{P}} h \, \mathrm{d}\mathcal{P},$$

$\mathbf{n}$ étant la normale sortante de $\partial\mathcal{P}$. On en déduit le *bilan de l'énergie* sous forme locale :

$$\dot{\epsilon} = -\mathrm{div}\,\mathbf{q} + h \quad \text{dans } \Omega. \tag{2.1}$$

L'inégalité de Clausius-Duhem s'écrit

$$\dot{S}(\mathcal{P}) \geqslant D(\mathcal{P}) \quad \text{pour toute partie } \mathcal{P} \subset \Omega,$$

où

$$S(\mathcal{P}) \coloneqq \int_{\mathcal{P}} s \, \mathrm{d}\mathcal{P}$$

est l'*entropie* de la partie $\mathcal{P}$, définie à partir de sa densité volumique $s\colon \Omega \times \mathbb{R}_+^* \to \mathbb{R}$, et

$$D(\mathcal{P}) \coloneqq -\int_{\partial\mathcal{P}} T^{-1}\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S + \int_{\mathcal{P}} T^{-1}h\,\mathrm{d}\mathcal{P}$$

est la *dissipation thermique*, avec $T > 0$ la *température absolue*. On en déduit l'*inégalité de Clausius-Duhem* sous forme locale :

$$\dot{s} \geqslant -\mathrm{div}(T^{-1}\mathbf{q}) + T^{-1}h \quad \text{dans } \Omega. \tag{2.2}$$

En multipliant (2.2) par T et en utilisant (2.1), on en déduit que

$$0 \leqslant T\dot{s} - (-\mathrm{div}\,\mathbf{q} + h) - T^{-1}\mathbf{q}\cdot\nabla T = T\dot{s} - \dot{\epsilon} - T^{-1}\mathbf{q}\cdot\nabla T.$$

En introduisant l'*énergie libre de Helmholtz*

$$\psi \coloneqq \epsilon - Ts, \tag{2.3}$$

l'inégalité précédente peut se réécrire comme

$$\dot{\psi} \leqslant -s\dot{T} - T^{-1}\mathbf{q}\cdot\nabla T \quad \text{dans } \Omega \tag{2.4}$$

On appelle cette dernière l'*inégalité de dissipation réduite*. Elle impose des restrictions sur le choix des lois de comportement qui caractérisent ψ, s et $\mathbf{q}$. En effet, si l'on suppose que

$$\psi = \psi(T, \nabla T), \quad s = s(T, \nabla T), \quad \mathbf{q} = \mathbf{q}(T, \nabla T), \tag{2.5}$$

en demandant, suivant le postulat de Coleman et Noll [16], que l'inégalité (2.4) soit vérifiée *pour tout processus admissible*, c'est à dire, pour tout couple $(T, \nabla T) \in \mathbb{R}_+^* \times \mathbb{R}^3$ et tout $(\dot{T}, \nabla\dot{T}) \in \mathbb{R}\times\mathbb{R}^3$, compte tenu des hypothèses constitutives (2.5), (2.4) se réécrit :

$$(\partial_T\psi + s)\dot{T} + \partial_{\nabla T}\psi\cdot\nabla\dot{T} + T^{-1}\mathbf{q}\cdot\nabla T \leqslant 0 \quad \text{dans } \Omega,$$

d'où l'on déduit que :

- l'énergie libre ψ est indépendante de ∇T :

$$\partial_{\nabla T}\psi = 0 \iff \psi = \psi(T);$$

- l'entropie s est, par conséquent, indépendante de ∇T aussi, et de plus

$$s = s(T) = -\psi'(T);$$

- le flux de chaleur $\mathbf{q}$ est tel que

$$\mathbf{q}(T, \nabla T)\cdot\nabla T \leqslant 0, \quad \forall (T, \nabla T) \in \mathbb{R}_+^* \times \mathbb{R}^3. \tag{2.6}$$

Proposition 2.1. *L'inégalité* (2.6) *implique l'existence (voir, e.g., [27]) d'un tenseur* $\mathbf{K}(T, \nabla T) \in \mathrm{Lin}$*, dit* tenseur de conductivité*, tel que*

$$\mathbf{q}(T, \nabla T) = -\mathbf{K}(T, \nabla T)\nabla T. \tag{2.7}$$

Démonstration. On introduit, pour tout $T > 0$ fixé, la fonction $\bar{\mathbf{q}}(\cdot) := \mathbf{q}(T, \cdot)$, puis les deux applications de classe C^1 suivantes :

$$\mathbb{R}^3 \ni \mathbf{u} \mapsto \bar{\mathbf{q}}(\mathbf{u}) \cdot \mathbf{u} =: a(\mathbf{u}) \in \mathbb{R},$$
$$[0, 1] \ni \alpha \mapsto \bar{\mathbf{q}}(\alpha \mathbf{u}) =: \mathbf{a}(\alpha) \in \mathbb{R}^3, \quad \forall \mathbf{u} \in \mathbb{R}^3 \text{ fixé.}$$

Puisque $a(\mathbf{0}) = 0$, et que $a(\mathbf{u}) \leqslant 0$ d'après (2.6), le point $\mathbf{u} = \mathbf{0}$ est un point critique (maximum global) de $a(\cdot)$, et on a la condition de stationnarité

$$\frac{\mathrm{d}}{\mathrm{d}\eta} a(\mathbf{0} + \eta \mathbf{h})|_{\eta=0} = 0, \quad \forall \mathbf{h} \in \mathbb{R}^3,$$

d'où $\bar{\mathbf{q}}(\mathbf{0}) = \mathbf{0}$. On a donc que *le flux de chaleur doit s'annuler lorsque le gradient de température est nul*. De plus, $\mathbf{a}(0) = \mathbf{0}$, d'où

$$\bar{\mathbf{q}}(\mathbf{u}) = \mathbf{a}(1) = \mathbf{a}(0) + \int_0^1 \mathbf{a}'(\alpha)\,\mathrm{d}\alpha = \left(\int_0^1 D\bar{\mathbf{q}}(\alpha \mathbf{u})\,\mathrm{d}\alpha \right) \mathbf{u},$$

$D\bar{\mathbf{q}}(\mathbf{v}) \in \mathrm{Lin}$ désignant la différentielle de $\bar{\mathbf{q}}$ en $\mathbf{v} \in \mathbb{R}^3$. Il suffit alors de poser

$$\mathbf{K}(T, \nabla T) := -\int_0^1 D\bar{\mathbf{q}}(\alpha \nabla T)\,\mathrm{d}\alpha$$

pour trouver la représentation (2.7). La condition (2.6) se réécrit alors :

$$\mathbf{K}(T, \nabla T) \cdot \nabla T \geqslant 0, \quad \forall (T, \nabla T) \in \mathbb{R}_+^* \times \mathbb{R}^3.$$

□

Avec le choix $\psi(T) = -\lambda T(\ln T - 1)$ pour l'énergie libre, $\lambda > 0$ étant la *chaleur spécifique*, on obtient alors $\epsilon(T) = \lambda T$ et $s(T) = \lambda \ln T$; de plus, en choisissant $\mathbf{K}(T, \nabla T) = \kappa \mathbf{I}$, avec $\kappa > 0$ la *conductivité* et $\mathbf{I}$ l'identité sur $\mathbb{R}^3$, le bilan de l'énergie (2.1) prend la forme usuelle de l'équation de la chaleur :

$$\lambda \dot{T} = \kappa \Delta T + h \quad \text{dans } \Omega.$$

Dans le cas d'un milieu *déformable*, l'inégalité de Clausius-Duhem a la même forme que dans le cas d'un milieu rigide ; en revanche, le bilan de l'énergie s'écrit

$$\dot{E}(\mathcal{P}) = Q(\mathcal{P}) + \Pi^s(\mathcal{P}) \text{ pour toute partie } \mathcal{P} \subset \Omega,$$
$$\Pi^s(\mathcal{P}) := \int_{\mathcal{P}} \boldsymbol{\sigma} : \mathbf{e}(\dot{\mathbf{u}})\,\mathrm{d}\mathcal{P},$$

$\boldsymbol{\sigma} : \overline{\Omega} \times \mathbb{R}_+ \to \mathrm{Sym}$ étant le tenseur des contraintes de Cauchy et $\mathbf{u} : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}^3$ le champ de déplacements ; sa version locale devient

$$\dot{\epsilon} = -\mathrm{div}\,\mathbf{q} + \boldsymbol{\sigma} : \mathbf{e}(\dot{\mathbf{u}}) + h \quad \text{dans } \Omega.$$

Par conséquent, la nouvelle forme de l'inégalité de dissipation réduite (2.4) est

$$\dot{\psi} \leqslant -s\dot{T} - T^{-1}\mathbf{q} \cdot \nabla T + \boldsymbol{\sigma} : \mathbf{e}(\dot{\mathbf{u}}) \quad \text{dans } \Omega,$$

avec le même choix (2.3) de l'énergie libre ψ. On peut ensuite appliquer les mêmes arguments que dans le cas d'un milieu rigide pour déduire les restrictions aux lois de comportement, où les quantités concernées sont, dans ce cas, ψ, s, $\mathbf{q}$ et le tenseur de Cauchy $\boldsymbol{\sigma}$.

Enfin, si l'on considère un milieu *thermo-piézoélectrique* (voir e.g. [13]), le bilan de l'énergie s'écrit

$$\dot{E}(\mathcal{P}) = Q(\mathcal{P}) + \Pi^s(\mathcal{P}) + \Pi^e(\mathcal{P}) \text{ pour toute partie } \mathcal{P} \subset \Omega,$$
$$\Pi^e(\mathcal{P}) := \int_{\mathcal{P}} \dot{\mathbf{D}} \cdot \mathbf{E} \, \mathrm{d}\mathcal{P},$$

$\mathbf{D}$ étant le déplacement électrique et $\mathbf{E}$ le champ électrique, d'où la version locale

$$\dot{\epsilon} = -\mathrm{div}\,\mathbf{q} + \boldsymbol{\sigma} : \mathbf{e}(\dot{\mathbf{u}}) + \dot{\mathbf{D}} \cdot \mathbf{E} + h \quad \text{dans } \Omega.$$

L'inégalité de Clausius-Duhem a toujours la forme (2.2). Dans ce cas, le rôle de l'énergie libre de Helmholtz ψ est joué par la fonction auxiliaire $G := \epsilon - Ts - \mathbf{D} \cdot \mathbf{E}$, appelée parfois *enthalpie électrique* dans la littérature.

2.2 Tenseur Acoustique et Propagation d'Ondes

On considère ici un milieu de densité $\rho = \rho(x)$ et *linéairement élastique*, c'est à dire qu'il est caractérisé par la loi de comportement

$$\boldsymbol{\sigma}(x,t) = \mathbf{C}(x)\mathbf{e}(\mathbf{u})(x,t), \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}_+,$$

où $\mathbf{C} = \mathbf{C}(x) \in \mathbb{L}\mathrm{in}$ est le *tenseur d'élasticité*.

La notion de tenseur acoustique est centrale dans l'étude de la propagation d'ondes dans un milieu linéairement élastique. Fixons $x \in \Omega$ et soit $\boldsymbol{\nu} \in \mathbb{S}^2$ un vecteur de norme unitaire. Puisque $\mathbf{C} \in \mathbb{L}\mathrm{in}$, l'application

$$\mathbb{R}^3 \ni \mathbf{a} \mapsto \rho^{-1}\mathbf{C}[\mathbf{a} \otimes \boldsymbol{\nu}]\boldsymbol{\nu} \in \mathbb{R}^3,$$

est un endomorphisme de $\mathbb{R}^3$, soit un tenseur du second ordre. On définit alors le *tenseur acoustique* $\mathbf{A}(\boldsymbol{\nu}) \in \mathrm{Lin}$ *associé à la direction* $\boldsymbol{\nu}$:

$$\mathbf{A}(\boldsymbol{\nu})\mathbf{a} := \rho^{-1}\mathbf{C}[\mathbf{a} \otimes \boldsymbol{\nu}]\boldsymbol{\nu}, \quad \forall \mathbf{a} \in \mathbb{R}^3. \tag{2.8}$$

En composantes, on a

$$A_{ik}(\boldsymbol{\nu}) = \rho^{-1} C_{ijk\ell} \nu_j \nu_i.$$

On rappelle deux propriétés de $\mathbf{A}(\boldsymbol{\nu})$ démontrées dans [33] :

— $\mathbf{A}(\boldsymbol{\nu})$ est symétrique pour tout $\boldsymbol{\nu} \in \mathbb{S}^2$ si et seulement si $\mathbf{C}$ est symétrique ;

— $\mathbf{A}(\boldsymbol{\nu})$ est défini positif pour tout $\boldsymbol{\nu} \in \mathbb{S}^2$ si et seulement si $\mathbf{C}$ est fortement elliptique.

Lorsque le milieu est constitué d'un matériau *isotrope*, avec modules de Lamé μ et λ, on peut montrer que $\mathbf{A}(\boldsymbol{\nu})$ admet la représentation

$$\mathbf{A}(\boldsymbol{\nu}) = c_1^2 \boldsymbol{\nu} \otimes \boldsymbol{\nu} + c_2^2 (\mathbf{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu}),$$

où c_1 et c_2 sont les *vitesses d'onde* :

$$c_1^2 = \frac{2\mu + \lambda}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}.$$

De plus, c_1^2 et c_2^2 sont les valeurs propres de $\mathbf{A}(\boldsymbol{\nu})$.

Pour mieux expliquer la définition (2.8), ainsi que la terminologie utilisée pour c_1 et c_2, considérons maintenant, sous l'hypothèse d'*homogénéité* du milieu, une *onde progressive* plane, c'est à dire, un champ de déplacements $\mathbf{u}\colon \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}^3$ de la forme

$$\mathbf{u}(x,t) = \mathbf{a}\,\varphi(\boldsymbol{\nu} \cdot \mathbf{x} - ct), \quad \mathbf{x} := x - o, \tag{2.9}$$

où :

(i) $\varphi\colon \mathbb{R} \to \mathbb{R}$ est une fonction de classe C^2, telle que $\varphi'' \not\equiv 0$;

(ii) $\mathbf{a}$, $\boldsymbol{\nu} \in \mathbb{S}^2$ sont deux vecteurs de norme unitaire, appelés respectivement *direction du mouvement* et *direction de propagation* ;

(iii) la quantité scalaire c est appelée *vitesse de propagation*.

L'onde est dite *longitudinale* lorsque $\mathbf{a} \times \boldsymbol{\nu} = \mathbf{0}$, *transversale* lorsque $\mathbf{a} \cdot \boldsymbol{\nu} = 0$, *élastique* si $\mathbf{u}$ donné par (2.9) satisfait l'équation du mouvement

$$\rho\ddot{\mathbf{u}} = \operatorname{div} \mathbf{C}[\nabla\mathbf{u}] \quad \text{dans } \Omega \times \mathbb{R}_+^*, \tag{2.10}$$

où les forces non-inertielles sont nulles. Compte tenu de l'expression (2.9), un calcul direct donne

$$\operatorname{div} \mathbf{C}[\nabla\mathbf{u}] = \varphi''\mathbf{C}[\mathbf{a} \otimes \boldsymbol{\nu}]\boldsymbol{\nu}, \quad \rho\ddot{\mathbf{u}} = \rho c^2 \varphi'' \mathbf{a},$$

de telle sorte que (2.10) prend la forme d'une équation aux valeurs propres :

$$\mathbf{A}(\boldsymbol{\nu})\mathbf{a} = c^2\mathbf{a}. \tag{2.11}$$

Ainsi, pour toute direction fixée $\boldsymbol{\nu} \in \mathbb{S}^2$, si une onde progressive de la forme (2.9) se propage dans le milieu, la relation (2.11), appelée *condition de propagation de Fresnel-Hadamard*, est satisfaite. Cela signifie donc que l'amplitude de l'onde doit être un vecteur propre de $\mathbf{A}(\boldsymbol{\nu})$, et que le carré de la vitesse de propagation doit être la valeur propre associée.

Deuxième partie

Modélisation Mathématique

Chapitre 3

Problèmes Dynamique et Quasi-Statique

Ce chapitre est consacré à la présentation du modèle mathématique, des problèmes dynamique et quasi-statique, et à la démonstration d'existence et unicité des solutions respectives. Les deux problèmes sont posés, *a priori*, sur un domaine tridimensionnel de forme arbitraire, borné, connexe et avec frontière Lipschitz-continue. Nous traitons le problème dynamique dans le cadre de la théorie de Hille-Yosida, le problème quasi-statique avec la méthode de Faedo-Galerkin. Le passage du problème dynamique au problème quasi-statique est justifié par l'adimensionnalisation des équations de départ, qui fait apparaître l'influence d'un petit paramètre δ, le rapport entre la vitesse maximale de propagation d'une onde élastique dans le milieu et la vitesse de la lumière. Nous donnons aussi un aperçu sur l'étude de la convergence de la solution du problème dynamique vers celle du problème quasi-statique lorsque $\delta \to 0$, en exprimant les données initiales concernant les champs électrique et magnétique adimensionnalisés en fonction des données initiales correspondantes du problème non adimensionnalisé.

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MODELING OF SMART MATERIALS WITH THERMAL EFFECTS: DYNAMIC AND QUASI-STATIC EVOLUTION

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We present a mathematical model for linear magneto-electro-thermo-elastic continua, as sensors and actuators can be thought of, and prove the well-posedness of the dynamic and quasi-static problems. The two proofs are accomplished, respectively, by means of the Hille-Yosida theory and of the Faedo-Galerkin method. A validation of the quasi-static hypothesis is provided by a nondimensionalization of the dynamic problem equations. We also hint at the study of the convergence of the solution to the dynamic problem to that to the quasi-static problem as a small parameter – the ratio of the largest propagation speed for an elastic wave in the body to the speed of light – tends to zero.

Keywords: piezoelectricity; magnetostriction; pyroelectricity; pyromagnetism; smart structures; sensors; actuators; nondimensionalization; multiscale problems; semigroups.

Introduction

When an elastic structure is subjected to a system of external loads, it undergoes a passive deformation. In the case of the so-called *smart structures*, the strain state is constantly under control by means of sensors and actuators, usually made of piezoelectric and/or magnetostrictive materials and integrated within the structure. In this paper, we consider magneto-electro-thermo-elastic materials as *smart materials*. The mechanical coupling of piezoelectric and magnetostrictive components in such structures gives rise to the so-called *magnetoelectric* effect, which is not present in the individual components. Typical geometries where such an effect may take place are given, for instance, by multilayer composites [20, 21], by structures made up of a homogeneous matrix within which particles of various form (mostly ellipsoidal) are dispersed [12, 13], or even fibrous materials, where parallel cylinders are inserted into the homogeneous matrix [1]. For a detailed description of the couplings and the multiphysics phenomena occurring in such structures, as well as of their applications, see, e.g., references [2], [9], [16], [17], [23] and [26]. In most literature, the coupling between mechanical and magnetic effects is expressed by a *linear* constitutive equation, and thus the word 'magnetostrictive' is replaced by *piezomagnetic*; we will adopt the same point of view and convention in the sequel.

The goal of this work is to enrich the classic models of piezoelectric and piezomagnetic sensors, by taking account of the temperature influence, which in some cases cannot be neglected; for instance, the effects of *pyroelectricity* and *pyromagnetism* may be relevant for what concerns energy harvesting performances [21]. To this purpose, it is necessary to add one further equation to the model, i.e., the energy balance. A distinctive feature of the problems encountered in applications is the presence of several parameters, which show the coexistence of different scales: for instance, the thickness of the piezoelectric/piezomagnetic layer may be small with respect to the other dimensions of the structure, the temperature influence may be relevant only on certain unknowns or on certain parts of the multi-structure, etc. In most situations, the superimposition of two wave propagation phenomena characterized by completely different velocities, as is the case with elastic and electromagnetic waves, entails an unworkable numerical treatment of the problem. This issue can be addressed by resorting to a *quasi-static model*, where the expression 'quasi-static' refers to the assumption that *the electric and magnetic fields can be expressed as gradients of the corresponding potentials*. This assumption is justified by means of a nondimensionalization procedure, carried out on the equations of the problem, which points out the influence of a *small parameter* δ – namely, the ratio between the largest propagation speed for an elastic wave in the body and the speed of light. Such a procedure was performed in [18] for a piezoelectric material without considering the temperature effects; an *a priori* quasi-static assumption was made, e.g., in [4] and [25] in the case of a thermo-piezoelectric material, whereas the same hypothesis for a magneto-electro-thermo-elastic material was made in [1] and [3].

In this paper, after discussing modeling aspects, attention is focused on the well-posedness of the problem in its most general setting – referred to as *dynamic* problem in the sequel – whose proof is accomplished by virtue of the Hille-Yosida theory (Section 1). In Section 2, a formal nondimensionalization of the equations is performed, so as (*i*) to extend the results by [18] and (*ii*) to justify the quasi-static assumption of [1], [3], [4] and [25]; then, the well-posedness of the quasi-static problem is obtained by virtue of the Faedo-Galerkin method, along the lines of Lions and Miara [24, 25]. Finally, we provide another justification of the convergence of the solution to the dynamic problem to that of the quasi-static problem as $\delta \to 0$. We conclude with a discussion about the rigorous mathematical justification of this convergence and an overview about addressed and unaddressed problems related to the mathematical modeling of smart materials. Typical numerical values of the material parameters involved in the problem are listed in Table 1 of the Appendix. These values have been obtained from [22] taking into account the corrections pointed out by [14], [21] and [32] and adding an estimation of the calorific capacity from [20] and of the thermal conductivity from [27].

Notation

In what follows, we always assume a cartesian frame to have been fixed once and for all in the usual three-dimensional euclidean point space. Thus, we identify with (and denote by) $\mathbb{R}^3$ both this space and its associated translation vector space. Throughout the paper, $\Omega \subset \mathbb{R}^3$ denotes an open, bounded, connected region, with Lipschitz-continuous boundary $\partial\Omega$, occupied by a continuum made of magneto-electro-thermo-elastic material in its reference configuration. The typical point of Ω is denoted by x and time by t. A function and its typical value are denoted by the same letter. Time derivative of (scalar, vector or tensor) field Φ is denoted by either $\dot{\Phi}$ or $\partial_t \Phi$ [a]. Scalars are denoted by light-face letters, vector and tensor fields of any order by bold-face letters. The word 'tensor' is used as a synonym of 'linear transformation between vector spaces', as is customary in continuum mechanics. Unless noted otherwise, the matrix representation of tensor $\mathbf{A}$ (with respect to the fixed cartesian base) is denoted by $[\mathbf{A}]$. The scalar product of tensors $\mathbf{A}$ and $\mathbf{B}$ is denoted[b] by $\mathbf{A} : \mathbf{B}$, of vectors $\mathbf{a}$ and $\mathbf{b}$ by $\mathbf{a} \cdot \mathbf{b}$, the cross product by $\mathbf{a} \times \mathbf{b}$, and the euclidean norm of $\mathbf{a}$ by $|\mathbf{a}|$. The symmetric part of second-order tensor $\mathbf{A}$ is denoted by $\operatorname{sym}\mathbf{A}$, the linear space of symmetric second-order tensors by Sym. At times, we also make use of Einstein's summation convention, whereby the summation symbol is suppressed and summation over all possible values of an index is signaled implicitly by the fact that it occurs twice in a monomial term. The following notations are also used:

$$\mathbf{L}^2(\Omega) := [L^2(\Omega)]^k \text{ for } k = 3, 6, \quad \mathbf{H}^1(\Omega) := [H^1(\Omega)]^3,$$
$$\mathbf{L}^2(\Gamma) := [L^2(\Gamma)]^3 \text{ for } \Gamma \subset \partial\Omega, \quad \mathbf{H}(\mathrm{curl}, \Omega) := \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) : \nabla \times \mathbf{v} \in \mathbf{L}^2(\Omega) \right\},$$

where $H^1(\Omega)$ is the usual Hilbert-Sobolev space[c].

1. Dynamic Problem

The system of field equations we resort to consists of the point-wise momentum balance equation for three-dimensional continua, Maxwell's equations and the energy balance equation, in its version adapted to deformable electromagnetic materials (see, e.g., [10] for the case of a thermo-piezoelectric material):

$$\begin{cases} \rho\ddot{\mathbf{u}} - \operatorname{div}\boldsymbol{\sigma} = \mathbf{f} & x \in \Omega,\ t > 0, \\ \operatorname{div}\mathbf{D} = \rho_e & x \in \Omega,\ t > 0, \\ \operatorname{div}\mathbf{B} = 0 & x \in \Omega,\ t > 0, \\ \dot{\mathbf{D}} - \nabla \times \mathbf{H} = -\mathbf{J} & x \in \Omega,\ t > 0, \\ \dot{\mathbf{B}} + \nabla \times \mathbf{E} = \mathbf{0} & x \in \Omega,\ t > 0, \\ \dot{\epsilon} + \operatorname{div}\mathbf{q} - \boldsymbol{\sigma} : \dot{\mathbf{e}} - \dot{\mathbf{D}} \cdot \mathbf{E} - \dot{\mathbf{B}} \cdot \mathbf{H} = h & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where ρ is the mass density, ρ_e the free electric charge volume density, $\boldsymbol{\sigma}$ the Cauchy stress tensor, $\mathbf{D}$ the electric displacement, $\mathbf{B}$ the magnetic induction, ϵ the internal energy

[a] All three denote a partial derivative, as all formulations hold in a fixed reference domain.
[b] By definition, $\mathbf{A} : \mathbf{B} := \operatorname{tr}(\mathbf{A}\mathbf{B}^T) = \sum_{i,j=1}^{3} A_{ij} B_{ij}$ for $\mathbf{A}$ and $\mathbf{B}$ arbitrary second-order tensors.
[c] See [6] and, in particular, [7] and [15] for a detailed description of $\mathbf{H}(\mathrm{curl}, \Omega)$.

per unit volume, $\mathbf{q}$ the heat influx, $\mathbf{f}$ the body force, $\mathbf{J}$ an external current density, h an external heat supply, $\mathbf{e} = \mathbf{e}(\mathbf{u}) = \operatorname{sym} \nabla \mathbf{u}$ the strain tensor, $\mathbf{u}$ the displacement field, $\mathbf{E}$ the electric field and $\mathbf{H}$ the magnetic field. Equations $(1.1)_2$ to $(1.1)_5$ are usually referred to as, respectively, Gauss's law, Gauss's law for magnetism, Ampère's circuital law and Faraday's law of induction. We explicitly remark that ρ_e and $\mathbf{J}$ obey the following *continuity equation* (conservation of electric charge):

$$\dot{\rho}_e + \operatorname{div} \mathbf{J} = 0, \quad x \in \Omega, \ t > 0. \tag{1.2}$$

Boundary and initial conditions will be detailed later.

1.1. *Constitutive Assumptions*

The model presented here is formulated in terms of four unknowns: the displacement field $\mathbf{u}$, the electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$ and the absolute temperature T. However, in most situations, it is more convenient to replace T by the temperature variation θ with respect to a reference value T_0. In this section we introduce the *linear* coupled constitutive equations relating the set of *state quantities* $(\mathbf{e}, \mathbf{E}, \mathbf{H}, \theta)$, with $\mathbf{e} = \operatorname{sym} \nabla \mathbf{u}$, to the corresponding set of *dual quantities* $(\boldsymbol{\sigma}, \mathbf{D}, \mathbf{B}, s)$ where s is the entropy per unit volume, and show that these equations are consistent with continuum thermodynamics (see, e.g., [8] or [10]).

The point-wise version of the Second Principle of Thermodynamics (*entropy imbalance*) reads

$$\dot{s} \geqslant -\operatorname{div}(T^{-1}\mathbf{q}) + T^{-1}h, \quad x \in \Omega, \ t > 0, \tag{1.3}$$

with $T > 0$ the absolute temperature. Upon introducing the *electromagnetic enthalpy*

$$G := \epsilon - Ts - \mathbf{D} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{H} \tag{1.4}$$

it is easy to arrive at the following version of the entropy imbalance, which does not involve the heat supply h:

$$\dot{G} + s\dot{T} - \boldsymbol{\sigma} : \dot{\mathbf{e}} + \mathbf{D} \cdot \dot{\mathbf{E}} + \mathbf{B} \cdot \dot{\mathbf{H}} + T^{-1}\mathbf{q} \cdot \nabla T \leqslant 0; \tag{1.5}$$

for this reason, (1.5) is often referred to as *reduced dissipation inequality*. The inequality suggests that the quantities in the need of constitutive specifications are electromagnetic enthalpy, entropy, stress, electric displacement, magnetic induction and heat influx. We make the following constitutive assumptions:

$$\begin{aligned} &G = G(\mathbf{e}, \mathbf{E}, \mathbf{H}, T, \nabla T), \quad s = s(\mathbf{e}, \mathbf{E}, \mathbf{H}, T, \nabla T), \quad \mathbf{q} = \mathbf{q}(\mathbf{e}, \mathbf{E}, \mathbf{H}, T, \nabla T), \\ &\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{e}, \mathbf{E}, \mathbf{H}, T, \nabla T), \quad \mathbf{D} = \mathbf{D}(\mathbf{e}, \mathbf{E}, \mathbf{H}, T, \nabla T), \quad \mathbf{B} = \mathbf{B}(\mathbf{e}, \mathbf{E}, \mathbf{H}, T, \nabla T), \end{aligned}$$

with which (1.5) takes the form

$$\begin{aligned} \partial_{\nabla T} G \cdot \nabla \dot{T} + (\partial_{\mathbf{e}} G - \boldsymbol{\sigma}) : \dot{\mathbf{e}} + (\partial_T G + s)\dot{T} + (\partial_{\mathbf{E}} G + \mathbf{D}) \cdot \dot{\mathbf{E}} + \\ + (\partial_{\mathbf{H}} G + \mathbf{B}) \cdot \dot{\mathbf{H}} + T^{-1}\mathbf{q} \cdot \nabla T \leqslant 0 \end{aligned} \tag{1.6}$$

and require, as in [11], that (1.6) *be satisfied whatever the local continuation of any conceivable process*; that is, on defining the *state* $\Theta := (\mathbf{e}, \mathbf{E}, \mathbf{H}, T, \nabla T) \in \mathrm{Sym} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3$, (1.6) must hold whatever $\dot{\Theta}$ at whatever state Θ. It follows that:

$$\begin{gathered} \partial_{\nabla T} G = \mathbf{0} \Longleftrightarrow G = G(\mathbf{e}, \mathbf{E}, \mathbf{H}, T), \\ s(\mathbf{e}, \mathbf{E}, \mathbf{H}, T) = -\partial_T G(\mathbf{e}, \mathbf{E}, \mathbf{H}, T), \qquad \boldsymbol{\sigma}(\mathbf{e}, \mathbf{E}, \mathbf{H}, T) = \partial_{\mathbf{e}} G(\mathbf{e}, \mathbf{E}, \mathbf{H}, T), \\ \mathbf{D}(\mathbf{e}, \mathbf{E}, \mathbf{H}, T) = -\partial_{\mathbf{E}} G(\mathbf{e}, \mathbf{E}, \mathbf{H}, T), \qquad \mathbf{B}(\mathbf{e}, \mathbf{E}, \mathbf{H}, T) = -\partial_{\mathbf{H}} G(\mathbf{e}, \mathbf{E}, \mathbf{H}, T), \\ \mathbf{q}(\mathbf{e}, \mathbf{E}, \mathbf{H}, T, \nabla T) \cdot \nabla T \leqslant 0. \end{gathered} \tag{1.7}$$

We have now to assign the expression of G in terms of the state quantities. With a view toward getting *linear constitutive equations*, we introduce the *temperature variation* θ, by writing

$$\begin{gathered} T(x,t) = T_0 + \theta(x,t), \quad x \in \Omega,\ t > 0, \\ \sup_{x \in \Omega} \theta(x,t)/T_0 \ll 1 \quad \forall t > 0, \end{gathered}$$

with $T_0 > 0$ the (constant) reference temperature of the body; we replace T by θ in the list of state quantities and choose[d]

$$\begin{aligned} G = G(\mathbf{e}, \mathbf{E}, \mathbf{H}, \theta) = \left(\frac{1}{2}\mathbf{C}\mathbf{e} - \mathbf{P}^T\mathbf{E} - \mathbf{R}^T\mathbf{H} - \boldsymbol{\beta}\theta \right) : \mathbf{e} - \frac{1}{2}\mathbf{X}\mathbf{E} \cdot \mathbf{E} - \frac{1}{2}\mathbf{M}\mathbf{H} \cdot \mathbf{H} + \\ - \boldsymbol{\alpha}\mathbf{E} \cdot \mathbf{H} - (\mathbf{p} \cdot \mathbf{E} + \mathbf{m} \cdot \mathbf{H})\theta - \frac{1}{2} c_v \theta^2. \end{aligned} \tag{1.8}$$

Let $\widetilde{\Theta} := (\mathbf{e}, \mathbf{E}, \mathbf{H}, \theta) \in \mathrm{Sym} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$. It follows from (1.7) that (see, e.g., [3], [21], [33]):

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}(\widetilde{\Theta}) = \mathbf{C}\mathbf{e} - \mathbf{P}^T\mathbf{E} - \mathbf{R}^T\mathbf{H} - \boldsymbol{\beta}\theta, \\ \mathbf{D} &= \mathbf{D}(\widetilde{\Theta}) = \mathbf{P}\mathbf{e} + \mathbf{X}\mathbf{E} + \boldsymbol{\alpha}\mathbf{H} + \mathbf{p}\theta, \\ \mathbf{B} &= \mathbf{B}(\widetilde{\Theta}) = \mathbf{R}\mathbf{e} + \boldsymbol{\alpha}\mathbf{E} + \mathbf{M}\mathbf{H} + \mathbf{m}\theta, \\ s &= s(\widetilde{\Theta}) = \boldsymbol{\beta} : \mathbf{e} + \mathbf{p} \cdot \mathbf{E} + \mathbf{m} \cdot \mathbf{H} + c_v\theta. \end{aligned} \tag{1.9}$$

In constitutive equations (1.9), $\mathbf{C} = (C_{ijk\ell})$, $\mathbf{P} = (P_{ijk})$, $\mathbf{R} = (R_{ijk})$, $\mathbf{X} = (X_{ij})$, $\mathbf{M} = (M_{ij})$, $\boldsymbol{\beta} = (\beta_{ij})$, $\boldsymbol{\alpha} = (\alpha_{ij})$, $\mathbf{p} = (p_i)$, $\mathbf{m} = (m_i)$ and c_v represent, respectively, the elasticity tensor, the piezoelectric tensor, the piezomagnetic tensor, the dielectric permittivity tensor, the magnetic permeability tensor, the thermal stress tensor, the magneto-electric tensor, the pyroelectric vector, the pyromagnetic vector, and the calorific capacity, defined such that $c_v T_0$ be the specific heat per unit volume of the material. Moreover, it can be shown that, whatever the values of the other state variables, the heat influx is null as long as the temperature gradient is; thus, mimicking the constitutive assumption of classic heat conduction, we can generally represent $\mathbf{q}$ as

$$\mathbf{q} = \mathbf{q}(\widetilde{\Theta}, \nabla\theta) = -\mathbf{K}(\widetilde{\Theta}, \nabla\theta)\nabla\theta,$$

[d]Physical meaning and hypotheses on the constitutive parameters are set forth hereinafter.

with $\mathbf{K} = \mathbf{K}(\widetilde{\Theta}, \nabla\theta)$ a second-order tensor. In order to satisfy $(1.7)_6$, $\mathbf{K}$ must obey the following general condition, for any fixed $\widetilde{\Theta}$:

$$\mathbf{K}\nabla\theta \cdot \nabla\theta \geqslant 0, \quad \forall\, \nabla\theta \in \mathbb{R}^3.$$

In the sequel, we shall consider $\mathbf{K}$ independent of the state variables. Hence, in addition to constitutive relationships (1.9), we have *Fourier's law*:

$$\mathbf{q} = \mathbf{q}(\nabla\theta) = -\mathbf{K}\nabla\theta. \tag{1.10}$$

In (1.10), $\mathbf{K} = (K_{ij})$ is the thermal conductivity tensor.

1.2. *Assumptions on the Material Parameters*

The symmetry and positivity conditions we require to be satisfied by density and constitutive parameters are listed below.

- The density ρ is positive:

$$\rho > 0, \quad \rho \in L^\infty(\Omega). \tag{1.11}$$

- The fourth-order elasticity tensor $\mathbf{C} = (C_{ijk\ell})$ is symmetric and positive definite:

$$\begin{aligned} &C_{ijk\ell} = C_{jik\ell} = C_{k\ell ij} = C_{ij\ell k}, \quad C_{ijk\ell} \in L^\infty(\Omega), \\ &C_{ijk\ell} b_{k\ell} b_{ij} \geqslant C \textstyle\sum_{i,j} |b_{ij}|^2, \ \text{ for all } b_{ij} = b_{ji} \in \mathbb{R},\ C > 0. \end{aligned}$$

- The third-order piezoelectric tensor[e] $\mathbf{P} = (P_{ijk})$ is symmetric with respect to the two last indices:

$$P_{ijk} = P_{ikj}, \quad P_{ijk} \in L^\infty(\Omega).$$

- The third-order piezomagnetic tensor $\mathbf{R} = (R_{ijk})$ is symmetric with respect to the two last indices:

$$R_{ijk} = R_{ikj}, \quad R_{ijk} \in L^\infty(\Omega).$$

- The second-order dielectric permittivity tensor $\mathbf{X} = (X_{ij})$ is symmetric and positive definite:

$$\begin{aligned} &X_{ij} = X_{ji}, \quad X_{ij} \in L^\infty(\Omega), \\ &X_{ij} a_j a_i \geqslant \mathcal{X} \textstyle\sum_i |a_i|^2, \ \text{ for all } a_i \in \mathbb{R},\ \mathcal{X} > 0. \end{aligned}$$

- The second-order magnetic permeability tensor $\mathbf{M} = (M_{ij})$ is symmetric and positive definite:

$$\begin{aligned} &M_{ij} = M_{ji}, \quad M_{ij} \in L^\infty(\Omega), \\ &M_{ij} a_j a_i \geqslant \mu \textstyle\sum_i |a_i|^2, \ \text{ for all } a_i \in \mathbb{R},\ \mu > 0. \end{aligned} \tag{1.12}$$

[e]By 'third-order tensor' we mean a linear transformation of the vector space of all second-order tensors into $\mathbb{R}^3$. For $\mathbf{P}$ a third-order tensor, its transpose $\mathbf{P}^T$ maps $\mathbb{R}^3$ onto the space of second-order tensors. In cartesian components, the following identity holds:

$$P^T_{ijk} = P_{kij}.$$

- The second-order thermal stress tensor $\boldsymbol{\beta} = (\beta_{ij})$ is symmetric:

$$\beta_{ij} = \beta_{ji}, \quad \beta_{ij} \in L^\infty(\Omega). \tag{1.13}$$

- The second-order magneto-electric tensor $\boldsymbol{\alpha} = (\alpha_{ij})$ is symmetric:

$$\alpha_{ij} = \alpha_{ji}, \quad \alpha_{ij} \in L^\infty(\Omega).$$

- The pyroelectric vector $\mathbf{p} = (p_i)$ is such that

$$p_i \in L^\infty(\Omega). \tag{1.14}$$

- The pyromagnetic vector $\mathbf{m} = (m_i)$ is such that

$$m_i \in L^\infty(\Omega). \tag{1.15}$$

- The calorific capacity c_v is positive:

$$c_v > 0, \quad c_v \in L^\infty(\Omega). \tag{1.16}$$

- The second-order thermal conductivity tensor $\mathbf{K} = (K_{ij})$ is symmetric and positive definite:

$$\begin{aligned} &K_{ij} = K_{ji}, \quad K_{ij} \in L^\infty(\Omega),\\ &K_{ij}a_j a_i \geqslant K \textstyle\sum_i |a_i|^2, \ \text{for all } a_i \in \mathbb{R},\ K > 0. \end{aligned}$$

- The following symmetric matrix, referred to in the sequel as *coupling matrix* (see [25])

$$[\mathbb{M}^c] := \begin{pmatrix} [\mathbf{X}] & [\boldsymbol{\alpha}] & [\mathbf{p}] \\ [\boldsymbol{\alpha}] & [\mathbf{M}] & [\mathbf{m}] \\ [\mathbf{p}]^T & [\mathbf{m}]^T & c_v \end{pmatrix}$$

is positive definite, i.e., there exists a constant $\mathscr{M} > 0$ such that

$$[\mathbb{M}^c]\mathbf{x} \cdot \mathbf{x} \geqslant \mathscr{M}|\mathbf{x}|^2, \quad \forall \mathbf{x} \in \mathbb{R}^7 \equiv \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R},$$

or, more explicitly,

$$\begin{aligned} &X_{ij}a_j a_i + M_{ij}b_j b_i + 2\alpha_{ij}a_j b_i + 2(p_k a_k)d + 2(m_k a_k)d + c_v d^2 \geqslant \\ &\geqslant C(a_i a_i + b_i b_i + d^2), \ \text{for all } a_i, b_i, d \in \mathbb{R}. \end{aligned} \tag{1.17}$$

1.3. *Field and Boundary Equations*

Before making explicit the complete system of governing equations, we make some remarks on the energy balance equation. From (1.4) and (1.7), it follows that

$$\dot{\epsilon} = \boldsymbol{\sigma} : \dot{\mathbf{e}} + T\dot{s} + \dot{\mathbf{D}} \cdot \mathbf{E} + \dot{\mathbf{B}} \cdot \mathbf{H};$$

with which the energy balance $(1.1)_6$ takes the form

$$T\dot{s} + \operatorname{div}\mathbf{q} = h;$$

by $(1.9)_4$ and (1.10), the last equation reads

$$T\left(c_v\dot{\theta} + \boldsymbol{\beta} : \dot{\mathbf{e}} + \mathbf{p} \cdot \dot{\mathbf{E}} + \mathbf{m} \cdot \dot{\mathbf{H}}\right) - \operatorname{div}\mathbf{K}\nabla\theta = h,$$

where $T = T_0 + \theta$. The left-hand side of this equation contains nonlinear terms (products of θ and time derivatives of the unknowns); upon neglecting them and setting $r \coloneqq h/T_0$, we obtain the following linearized version of the energy balance:

$$c_v\dot{\theta} + \boldsymbol{\beta} : \dot{\mathbf{e}} + \mathbf{p}\cdot\dot{\mathbf{E}} + \mathbf{m}\cdot\dot{\mathbf{H}} - \frac{1}{T_0}\operatorname{div}\mathbf{K}\nabla\theta = r. \tag{1.18}$$

All in all, by virtue of (1.9) and (1.10), the system of field equations (1.1) in the list of unknowns $\widetilde{\mathcal{U}} \coloneqq (\mathbf{u}, \mathbf{E}, \mathbf{H}, \theta)$ becomes[f]:

$$\begin{cases} \rho\ddot{\mathbf{u}} - \operatorname{div}\mathbf{Ce}(\mathbf{u}) + \operatorname{div}\mathbf{P}^T\mathbf{E} + \operatorname{div}\mathbf{R}^T\mathbf{H} + \operatorname{div}\boldsymbol{\beta}\theta = \mathbf{f} & x \in \Omega, \quad t > 0, \\ \operatorname{div}(\mathbf{Pe}(\mathbf{u}) + \mathbf{XE} + \boldsymbol{\alpha}\mathbf{H} + \mathbf{p}\theta) = \rho_e & x \in \Omega, \quad t > 0, \\ \operatorname{div}(\mathbf{Re}(\mathbf{u}) + \boldsymbol{\alpha}\mathbf{E} + \mathbf{MH} + \mathbf{m}\theta) = 0 & x \in \Omega, \quad t > 0, \\ \mathbf{X}\dot{\mathbf{E}} + \mathbf{Pe}(\dot{\mathbf{u}}) + \boldsymbol{\alpha}\dot{\mathbf{H}} + \mathbf{p}\dot{\theta} - \nabla\times\mathbf{H} = -\mathbf{J} & x \in \Omega, \quad t > 0, \\ \mathbf{M}\dot{\mathbf{H}} + \mathbf{Re}(\dot{\mathbf{u}}) + \boldsymbol{\alpha}\dot{\mathbf{E}} + \mathbf{m}\dot{\theta} + \nabla\times\mathbf{E} = \mathbf{0} & x \in \Omega, \quad t > 0, \\ c_v\dot{\theta} + \boldsymbol{\beta} : \mathbf{e}(\dot{\mathbf{u}}) + \mathbf{p}\cdot\dot{\mathbf{E}} + \mathbf{m}\cdot\dot{\mathbf{H}} - \dfrac{1}{T_0}\operatorname{div}\mathbf{K}\nabla\theta = r & x \in \Omega, \quad t > 0. \end{cases} \tag{1.19}$$

Remark 1.1. Note that (1.19) contains twelve scalar equations in ten scalar unknowns. Nevertheless, as we shall show in subsection 1.4, equations $(1.19)_2$ and $(1.19)_3$ actually just play the role of *compatibility conditions* on initial and boundary data, as well as on the free electric charge density ρ_e.

This system is equipped with the following initial conditions, for any $x \in \Omega$:

$$\begin{gathered} \mathbf{E}(x,0) = \mathbf{E}_0(x), \ \mathbf{H}(x,0) = \mathbf{H}_0(x), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \ \dot{\mathbf{u}}(x,0) = \mathbf{u}_1(x), \ \theta(x,0) = \theta_0(x), \end{gathered} \tag{1.20}$$

and with suitable boundary conditions. In particular, let $t > 0$ and let $\mathbf{n}$ be the outer unit normal vector field on $\partial\Omega$. As in [34], we consider four time-independent partitions of $\partial\Omega$: $(\Gamma_{mD}, \Gamma_{mN})$, $(\Gamma_{eD}, \Gamma_{eN})$, $(\Gamma_{gD}, \Gamma_{gN})$ and $(\Gamma_{tD}, \Gamma_{tN})$ with Γ_{mD}, Γ_{eD} Γ_{gD} and Γ_{tD} of strictly positive surface measure. We assign boundary values pertaining to mechanical quantities on Γ_{mD} and Γ_{mN}, to electrical quantities on Γ_{eD} and Γ_{eN}, to magnetic quantities on Γ_{gD} and Γ_{gN} and to thermal quantities on Γ_{tD} and Γ_{tN}. Namely, we assume that the body is clamped along Γ_{mD} and subjected to surface traction $\mathbf{g}$ on Γ_{mN}. Next, it is subjected to a vanishing temperature variation along Γ_{tD} and to heat influx ϱ on Γ_{tN}. Furthermore, the body is in contact with a perfect conductor on Γ_{eD} and with an infinitely permeable medium on Γ_{gD}, whereas it carries an electric charge surface density d on Γ_{eN} and a magnetic charge surface density b on Γ_{gN}. Hence, for any $t > 0$, we have

$$\begin{cases} \boldsymbol{\sigma}(\widetilde{\mathcal{U}})\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{mN}, & \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{mD}, \\ \mathbf{D}(\widetilde{\mathcal{U}})\cdot\mathbf{n} = d \ \text{ on } \Gamma_{eN}, & \mathbf{E}\times\mathbf{n} = \mathbf{0} \ \text{ on } \Gamma_{eD}, \\ \mathbf{B}(\widetilde{\mathcal{U}})\cdot\mathbf{n} = b \ \text{ on } \Gamma_{gN}, & \mathbf{H}\times\mathbf{n} = \mathbf{0} \ \text{ on } \Gamma_{gD}, \\ -\mathbf{q}(\theta)\cdot\mathbf{n} = \varrho \ \text{ on } \Gamma_{tN}, & \theta = 0 \quad \text{on } \Gamma_{tD}. \end{cases} \tag{1.21}$$

[f]Henceforth, to point out the unknowns of the problem, we replace the list of state quantities $\widetilde{\Theta}$ by the list of unknowns $\widetilde{\mathcal{U}}$ (and $\nabla\theta$ by θ) in the constitutive relationships.

Along with boundary conditions, we introduce the following spaces:

$$\mathbf{H}^1_{mD}(\Omega) := \left\{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{mD}\right\},$$
$$H^1_{tD}(\Omega) := \left\{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{tD}\right\},$$

and use analogous definitions for $H^1_{eD}(\Omega)$ and $H^1_{gD}(\Omega)$ [g]; finally, let

$$\mathbf{H}_{eD}(\text{curl}, \Omega) := \{\mathbf{v} \in \mathbf{H}(\text{curl}, \Omega) : \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{eD}\}$$

and analogously for $\mathbf{H}_{gD}(\text{curl}, \Omega)$ (see, again, [7] and [15]).

1.4. *Existence and Uniqueness*

In order to establish the well-posedness of the problem, we work in the context of the *semigroup theory*; in particular, we resort to the Hille-Yosida theorem for abstract linear differential equations with a source term [6], whose statement is recalled in the appendix. The main results of this subsection are summarized in the following statement.

Theorem 1.1. *Let $T > 0$. Assume the initial and boundary values have the following regularity properties:*

$$(\mathbf{u}_0, \mathbf{u}_1, \mathbf{E}_0, \mathbf{H}_0, \theta_0) \in \mathbf{H}^1_{mD}(\Omega) \times \mathbf{H}^1_{mD}(\Omega) \times \mathbf{H}_{eD}(\text{curl}, \Omega) \times \mathbf{H}_{gD}(\text{curl}, \Omega) \times H^1_{tD}(\Omega),$$
$$\text{div}\big(\mathbf{Ce}(\mathbf{u}_0) - \mathbf{P}^T\mathbf{E}_0 - \mathbf{R}^T\mathbf{H}_0 - \boldsymbol{\beta}\theta_0\big) \in \mathbf{L}^2(\Omega), \quad \text{div}\,\mathbf{K}\nabla\theta_0 \in L^2(\Omega),$$
$$\mathbf{g} \in C^3([0,T]; \mathbf{L}^2(\Gamma_{mN})), \quad \varrho \in C^3([0,T]; L^2(\Gamma_{mN})),$$
$$d(\cdot, t) \in L^1(\Gamma_{eN}) \quad \text{and} \quad b(\cdot, t) \in L^1(\Gamma_{gN}) \quad \forall t \in [0,T].$$

Also, let the source terms be such that

$$\mathbf{f} \in C^1([0,T]; \mathbf{L}^2(\Omega)), \quad \rho_e \in H^1(0,T; L^2(\Omega)) \cap C^0([0,T]; L^2(\Omega)),$$
$$\mathbf{J} \in C^1([0,T]; \mathbf{L}^2(\Omega)), \quad r \in C^1([0,T]; L^2(\Omega)),$$

and let the following compatibility conditions *hold:*

$$\begin{aligned} &\int_{\Gamma_{eN}} d(\cdot, 0)\,\mathrm{d}\Gamma + \int_{\Gamma_{eD}} (\mathbf{Pe}(\mathbf{u}_0) + \mathbf{XE}_0 + \boldsymbol{\alpha}\mathbf{H}_0 + \mathbf{p}\theta_0) \cdot \mathbf{n}\,\mathrm{d}\Gamma = \int_\Omega \rho_e(\cdot, 0)\,\mathrm{d}\Omega, \\ &\int_{\Gamma_{gN}} b(\cdot, 0)\,\mathrm{d}\Gamma + \int_{\Gamma_{gD}} (\mathbf{Re}(\mathbf{u}_0) + \boldsymbol{\alpha}\mathbf{E}_0 + \mathbf{MH}_0 + \mathbf{m}\theta_0) \cdot \mathbf{n}\,\mathrm{d}\Gamma = 0. \end{aligned} \tag{1.22}$$

Finally, let the following electromagnetic complementarity hypothesis *hold:*

$$\Gamma_{gD} = \Gamma_{eN} \quad \textit{and} \quad \Gamma_{eD} = \Gamma_{gN}. \tag{1.23}$$

Then, problem (1.19)-(1.20)-(1.21) *admits a unique strong solution* $(\mathbf{u}, \mathbf{E}, \mathbf{H}, \theta)$ *satisfying*

$$\begin{cases} \mathbf{u} \in C^2([0,T]; \mathbf{L}^2(\Omega)) \cap C^1([0,T]; \mathbf{H}^1_{mD}(\Omega)), \\ \mathbf{E} \in C^1([0,T]; \mathbf{L}^2(\Omega)) \cap C^0([0,T]; \mathbf{H}_{eD}(\text{curl}, \Omega)), \\ \mathbf{H} \in C^1([0,T]; \mathbf{L}^2(\Omega)) \cap C^0([0,T]; \mathbf{H}_{gD}(\text{curl}, \Omega)), \\ \theta \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H^1_{tD}(\Omega)). \end{cases}$$

[g] In particular, spaces $H^1_{eD}(\Omega)$ and $H^1_{gD}(\Omega)$ will be employed later, in the formulation of the quasi-static problem.

Proof. The proof is subdivided into six steps.

Step 1. With a view toward applying the Hille-Yosida theorem to our case, we have first of all to reduce our system of field equations into a system of *evolution equations*. Thus, we rewrite system (1.19) disregarding time-independent Maxwell's equations $(1.19)_2$ and $(1.19)_3$ and boundary conditions $(1.21)_{2,1}$ and $(1.21)_{3,1}$; we shall come back to these equations later. Therefore, we have

$$\begin{cases} \rho\ddot{\mathbf{u}} - \operatorname{div}\mathbf{Ce}(\mathbf{u}) + \operatorname{div}\mathbf{P}^T\mathbf{E} + \operatorname{div}\mathbf{R}^T\mathbf{H} + \operatorname{div}\boldsymbol{\beta}\theta = \mathbf{f} & x\in\Omega, \quad t>0, \\ \mathbf{X}\dot{\mathbf{E}} + \mathbf{Pe}(\dot{\mathbf{u}}) + \boldsymbol{\alpha}\dot{\mathbf{H}} + \mathbf{p}\dot{\theta} - \nabla\times\mathbf{H} = -\mathbf{J} & x\in\Omega, \quad t>0, \\ \mathbf{M}\dot{\mathbf{H}} + \mathbf{Re}(\dot{\mathbf{u}}) + \boldsymbol{\alpha}\dot{\mathbf{E}} + \mathbf{m}\dot{\theta} + \nabla\times\mathbf{E} = \mathbf{0} & x\in\Omega, \quad t>0, \\ c_v\dot{\theta} + \boldsymbol{\beta} : \mathbf{e}(\dot{\mathbf{u}}) + \mathbf{p}\cdot\dot{\mathbf{E}} + \mathbf{m}\cdot\dot{\mathbf{H}} - \dfrac{1}{T_0}\operatorname{div}\mathbf{K}\nabla\theta = r & x\in\Omega, \quad t>0. \end{cases} \tag{1.24}$$

with boundary conditions as in (1.21), except $(1.21)_{2,1}$ and $(1.21)_{3,1}$.

Step 2. As we deal with non-homogeneous boundary conditions, we must introduce trace liftings of the boundary values concerning the displacement field $\mathbf{u}$ and the temperature variation θ. To do so, we resort to the following auxiliary problem – actually, to a *one-parameter family of static problems*, where the parameter is time:

For any $t \geqslant 0$, find $\widehat{\mathbf{u}}$ and $\widehat{\theta}$ such that[h]

$$\begin{cases} \operatorname{div}\left(\mathbf{Ce}(\widehat{\mathbf{u}}) - \boldsymbol{\beta}\widehat{\theta}\right) = \mathbf{0}, & x\in\Omega, \\ \boldsymbol{\beta} : \mathbf{e}(\widehat{\mathbf{u}}) - \dfrac{1}{T_0}\operatorname{div}\mathbf{K}\nabla\widehat{\theta} = 0, & x\in\Omega, \end{cases} \tag{1.25}$$

with boundary conditions:

$$\begin{cases} \left(\mathbf{Ce}(\widehat{\mathbf{u}}) - \boldsymbol{\beta}\widehat{\theta}\right)\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{mN}, & \widehat{\mathbf{u}} = \mathbf{0} \quad \text{on } \Gamma_{mD}, \\ \mathbf{K}\nabla\widehat{\theta}\cdot\mathbf{n} = \varrho \quad \text{on } \Gamma_{tN}, & \widehat{\theta} = 0 \quad \text{on } \Gamma_{tD}. \end{cases} \tag{1.26}$$

Constitutive parameters and boundary values are the same as in (1.19). By resorting to the Lax-Milgram theorem (see [19]) we see that, if boundary values are such that

$$\mathbf{g} \in C^3([0,T];\mathbf{L}^2(\Gamma_{mN})), \quad \varrho \in C^3([0,T];L^2(\Gamma_{mN})),$$

then problem (1.25)-(1.26) admits a unique solution $(\widehat{\mathbf{u}}, \widehat{\theta})$ with

$$\begin{gathered} \widehat{\mathbf{u}} \in C^3([0,T];\mathbf{H}^1_{mD}(\Omega)), \\ \widehat{\theta} \in C^3([0,T];H^1_{tD}(\Omega)) \cap C^3([0,T];H^{\Delta}_{\mathbf{K}}(\Omega)), \end{gathered} \tag{1.27}$$

with $H^{\Delta}_{\mathbf{K}}(\Omega) := \left\{\theta \in L^2(\Omega) : \operatorname{div}\mathbf{K}\nabla\theta \in L^2(\Omega)\right\}$[i]. On denoting $\widehat{\mathbf{v}} := \partial_t\widehat{\mathbf{u}}$, we have $\widehat{\mathbf{v}}|_{\Gamma_{mD}} = \mathbf{0}$. We use functions $\widehat{\mathbf{u}}$, $\widehat{\mathbf{v}}$ and $\widehat{\theta}$ as trace liftings of boundary values $(1.21)_1$

[h]Here we write $\widehat{\mathbf{u}}$ and $\widehat{\theta}$ in place of, respectively, $\widehat{\mathbf{u}}(\cdot,t)$ and $\widehat{\theta}(\cdot,t)$ for the sake of notation.
[i]That $\widehat{\theta} \in C^3([0,T];H^{\Delta}_{\mathbf{K}}(\Omega))$ follows from the fact that $\widehat{\mathbf{u}} \in C^3([0,T];[H^1_{mD}(\Omega)]^3)$ and from the second equation of (1.25).

and $(1.21)_4$. We shall come back to boundary conditions $(1.21)_{2,1}$ and $(1.21)_{3,1}$ later on.

Step 3. Let $\mathbf{v} := \partial_t \mathbf{u}$ and

$$\mathbb{H} := \mathbf{H}^1_{mD}(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \times L^2(\Omega).$$

Then we can define

$$U := \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{E} \\ \mathbf{H} \\ \theta \end{pmatrix}, \quad U : [0,T] \to \mathbb{H}.$$

We also introduce the *trace-lifting vector*

$$\widehat{U} := \begin{pmatrix} \widehat{\mathbf{u}} \\ \widehat{\mathbf{v}} \\ \mathbf{0} \\ \mathbf{0} \\ \widehat{\theta} \end{pmatrix} \qquad \text{and} \qquad \widetilde{U} := U - \widehat{U} = \begin{pmatrix} \widetilde{\mathbf{u}} \\ \widetilde{\mathbf{v}} \\ \mathbf{E} \\ \mathbf{H} \\ \widetilde{\theta} \end{pmatrix} \in \mathbb{H}.$$

We endow the Hilbert space $\mathbb{H}$ with the following scalar product:

$$\begin{aligned} \big(U_1, U_2\big)_{\mathbb{H}} := & \int_\Omega \mathbf{C}\mathbf{e}(\mathbf{u}_1) : \mathbf{e}(\mathbf{u}_2)\,\mathrm{d}\Omega + \int_\Omega \mathbf{v}_1 \cdot \mathbf{v}_2\,\mathrm{d}\Omega + \\ & + \int_\Omega \mathbf{E}_1 \cdot \mathbf{E}_2\,\mathrm{d}\Omega + \int_\Omega \mathbf{H}_1 \cdot \mathbf{H}_2\,\mathrm{d}\Omega + \int_\Omega \theta_1\theta_2\,\mathrm{d}\Omega, \end{aligned} \tag{1.28}$$

for all $U_1, U_2 \in \mathbb{H}$. Also, we define the domain $D(\mathcal{A})$ of the differential operator $\mathcal{A}$ which is introduced hereinafter:

$$\begin{aligned} D(\mathcal{A}) := \Big\{ U \in \mathbb{H} : \ & \operatorname{div}\big(\mathbf{C}\mathbf{e}(\mathbf{u}) - \mathbf{P}^T\mathbf{E} - \mathbf{R}^T\mathbf{H} - \boldsymbol{\beta}\theta\big) \in \mathbf{L}^2(\Omega),\ \mathbf{e}(\mathbf{v}) \in \mathbf{L}^2(\Omega), \\ & \nabla \times \mathbf{E} \in \mathbf{L}^2(\Omega),\ \nabla \times \mathbf{H} \in \mathbf{L}^2(\Omega),\ \operatorname{div} \mathbf{K}\nabla\theta \in L^2(\Omega), \\ & \big(\mathbf{C}\mathbf{e}(\mathbf{u}) - \mathbf{P}^T\mathbf{E} - \mathbf{R}^T\mathbf{H} - \boldsymbol{\beta}\theta\big)\mathbf{n} = \mathbf{0} \text{ and } \mathbf{K}\nabla\theta \cdot \mathbf{n} = 0 \text{ on } \Gamma_{tN}, \\ & \mathbf{u} = \mathbf{0} \text{ and } \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{mD},\ \theta = 0 \text{ on } \Gamma_{tD}, \\ & \mathbf{E} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{eD},\ \mathbf{H} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{gD} \Big\}. \end{aligned} \tag{1.29}$$

Step 4. We are now in a position to write system (1.24) in the form of a differential equation:

$$\mathcal{M}\frac{\mathrm{d}\widetilde{U}}{\mathrm{d}t} + \mathcal{A}\widetilde{U} = \widetilde{F}, \tag{1.30}$$

$$\widetilde{F} := F - \mathcal{A}\widehat{U} - \mathcal{M}\frac{\mathrm{d}\widehat{U}}{\mathrm{d}t}, \quad F := \begin{pmatrix} \mathbf{0} \\ \mathbf{f} \\ -\mathbf{J} \\ \mathbf{0} \\ r \end{pmatrix},$$

with

$$[\mathcal{M}] = \begin{pmatrix} [\mathbf{I}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & \rho[\mathbf{I}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{X}] & [\alpha] & [\mathbf{p}] \\ [\mathbf{0}] & [\mathbf{0}] & [\alpha] & [\mathbf{M}] & [\mathbf{m}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{p}]^T & [\mathbf{m}]^T & c_v \end{pmatrix}, \quad \mathcal{A}U := \begin{pmatrix} -\mathbf{v} \\ -\mathrm{div}\Big(\mathbf{Ce}(\mathbf{u}) - \mathbf{P}^T\mathbf{E} - \mathbf{R}^T\mathbf{H} - \boldsymbol{\beta}\theta\Big) \\ \mathbf{Pe}(\mathbf{v}) - \nabla\times\mathbf{H} \\ \mathbf{Re}(\mathbf{v}) + \nabla\times\mathbf{E} \\ \boldsymbol{\beta} : \mathbf{e}(\mathbf{v}) - \dfrac{1}{T_0}\mathrm{div}\,\mathbf{K}\nabla\theta \end{pmatrix}.$$

Note that, by (1.27), $\tilde{F} \in C^1([0,T];\mathbb{H})$ which is in accordance with the hypotheses of the Hille-Yosida theorem. By the hypotheses set forth in subsection 1.2, $[\mathcal{M}]$ is a symmetric and positive definite matrix, thus if $\mathcal{M}$ is regarded as an endomorphism of $\mathcal{V} := \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}$, it is a symmetric and positive definite linear operator. Therefore, $\mathcal{M}$ admits a unique square root $\mathcal{M}^{1/2}$, which is symmetric and positive definite as well. We can then set

$$\widetilde{V} := \mathcal{M}^{1/2}\widetilde{U},$$

so as to give (1.30) the form:

$$\frac{\mathrm{d}\widetilde{V}}{\mathrm{d}t} + \mathcal{B}\widetilde{V} = \overline{F}, \qquad \mathcal{B} := \mathcal{M}^{-1/2}\mathcal{A}\mathcal{M}^{-1/2},\ \overline{F} := \mathcal{M}^{-1/2}\widetilde{F}.$$

Let us remark that $\widetilde{V} \in D(\mathcal{B})$ if and only if $\widetilde{U} \in D(\mathcal{A})$, since $\mathcal{B}\widetilde{V} = \mathcal{M}^{-1/2}\mathcal{A}\widetilde{U}$ for any $\widetilde{V} = \mathcal{M}^{1/2}\widetilde{U}$. Also, it is easy to check that, if $\mathcal{M}$ is symmetric and positive definite as an endomorphism of $\mathcal{V}$ with respect to the natural scalar product in $\mathcal{V}$, then $\mathcal{M}$ is self-adjoint and positive definite as a linear operator of $\mathbb{H}$ into itself with respect to scalar product (1.28). Indeed, for any $U_1, U_2 \in \mathbb{H}$, by definition (1.28) it results

$$\big(\mathcal{M}U_1, U_2\big)_{\mathbb{H}} = \int_\Omega \mathbf{Ce}(\mathbf{u}_1) : \mathbf{e}(\mathbf{u}_2)\,\mathrm{d}\Omega + \int_\Omega \rho\mathbf{v}_1\cdot\mathbf{v}_2\,\mathrm{d}\Omega + \int_\Omega \mathbb{M}^c\mathbf{x}_1\cdot\mathbf{x}_2\,\mathrm{d}\Omega,$$

where in the last term, $\mathbf{x}_\gamma := (\mathbf{E}_\gamma, \mathbf{H}_\gamma, \theta_\gamma)^T$, $\gamma = 1,2$. Now, the last two terms correspond exactly to the submatrix of $[\mathcal{M}]$ obtained by eliminating the first row and the first column, and this submatrix is symmetric and definite positive since $[\mathcal{M}]$ is. By the hypotheses listed in subsection 1.2, $\mathbf{C}$ is symmetric and positive definite as well. Hence we have

$$\big(\mathcal{M}U_1, U_2\big)_{\mathbb{H}} = \big(U_1, \mathcal{M}U_2\big)_{\mathbb{H}}, \quad \forall U_1, U_2 \in \mathbb{H},$$
$$\big(\mathcal{M}U, U\big)_{\mathbb{H}} \geqslant \min\{\rho_-, \mathscr{M}, 1\}\,\|U\|^2_{\mathbb{H}},\ \ \forall U \in \mathbb{H},$$

where $\rho_- := \inf_\Omega \rho$ and $\mathscr{M}$ denotes the coercivity constant of $\mathbb{M}^c$ (see (1.17)).

Step 5. In order to show that the problem

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{V}}{\mathrm{d}t} + \mathcal{B}\widetilde{V} = \overline{F}, \quad t > 0 \\ \widetilde{V}(0) = \widetilde{V}_0 \end{cases} \tag{1.31}$$

is well-posed for any $\widetilde{V}_0 \in D(\mathcal{B})$, we have to prove that $\mathcal{B}$ is maximal monotone. Note that, since $\mathcal{M} : \mathbb{H} \to \mathbb{H}$ is a self-adjoint positive definite operator (as well as $\mathcal{M}^{-1/2}$), we have, for any $\widetilde{V} = \mathcal{M}^{1/2}\widetilde{U}$,

$$\big(\mathcal{B}\widetilde{V}, \widetilde{V}\big)_{\mathbb{H}} = \big(\mathcal{M}^{-1/2}\mathcal{A}\mathcal{M}^{-1/2}\widetilde{V}, \widetilde{V}\big)_{\mathbb{H}} = \big(\mathcal{A}\mathcal{M}^{-1/2}\widetilde{V}, \mathcal{M}^{-1/2}\widetilde{V}\big)_{\mathbb{H}} = \big(\mathcal{A}\widetilde{U}, \widetilde{U}\big)_{\mathbb{H}}.$$

Thus, $\mathcal{B}$ is monotone *if and only if* $\mathcal{A}$ is monotone.

Monotonicity. The proof reduces to showing that $\mathcal{A}$ is monotone. For any $\widetilde{U} \in D(\mathcal{A})$, after integration by parts, we find:

$$\begin{aligned}\big(\mathcal{A}\widetilde{U}, \widetilde{U}\big)_{\mathbb{H}} = &-\int_{\partial\Omega} \big(\mathbf{Ce}(\widetilde{\mathbf{u}}) - \mathbf{P}^T\mathbf{E} - \mathbf{R}^T\mathbf{H} - \boldsymbol{\beta}\widetilde{\theta}\big)\mathbf{n} \cdot \widetilde{\mathbf{v}}\,\mathrm{d}\Gamma - \frac{1}{T_0}\int_{\partial\Omega} (\mathbf{K}\nabla\widetilde{\theta} \cdot \mathbf{n})\widetilde{\theta}\,\mathrm{d}\Gamma \\ &+ \int_{\partial\Omega} \mathbf{E} \times \mathbf{H} \cdot \mathbf{n}\,\mathrm{d}\Gamma + \frac{1}{T_0}\int_{\Omega} \mathbf{K}\nabla\widetilde{\theta} \cdot \nabla\widetilde{\theta}\,\mathrm{d}\Omega.\end{aligned}$$

The first two integrals in the sum vanish, due to homogeneous boundary conditions; also, $1/T_0 \int_\Omega \mathbf{K}\nabla\widetilde{\theta} \cdot \nabla\widetilde{\theta}\,\mathrm{d}\Omega \geqslant 0$ since $\mathbf{K}$ is positive definite, thus:

$$\begin{aligned}\big(\mathcal{A}\widetilde{U}, \widetilde{U}\big)_{\mathbb{H}} \geqslant \int_{\partial\Omega} \mathbf{E} \times \mathbf{H} \cdot \mathbf{n}\,\mathrm{d}\Gamma &= \int_{\Gamma_{eD}} \mathbf{E} \times \mathbf{H} \cdot \mathbf{n}\,\mathrm{d}\Gamma + \int_{\Gamma_{eN}} \mathbf{E} \times \mathbf{H} \cdot \mathbf{n}\,\mathrm{d}\Gamma = \\ &= \int_{\Gamma_{eD}} \mathbf{E} \times \mathbf{H} \cdot \mathbf{n}\,\mathrm{d}\Gamma + \int_{\Gamma_{gD}} \mathbf{E} \times \mathbf{H} \cdot \mathbf{n}\,\mathrm{d}\Gamma,\end{aligned}$$

where the last equality holds by assumption (1.23). Taking into account the chain of identities $\mathbf{E} \times \mathbf{H} \cdot \mathbf{n} = -\mathbf{E} \times \mathbf{n} \cdot \mathbf{H} = \mathbf{H} \times \mathbf{n} \cdot \mathbf{E}$ and boundary conditions $(1.21)_{2,2}$ and $(1.21)_{3,2}$, the right-hand side of the last inequality vanishes as well, hence:

$$\forall\widetilde{U} \in D(\mathcal{A}), \quad \big(\mathcal{A}\widetilde{U}, \widetilde{U}\big)_{\mathbb{H}} \geqslant 0,$$

i.e., $\mathcal{A}$ is monotone and so is $\mathcal{B}$.

Maximality. The operator $\mathcal{B} + \mathcal{I}$, with $\mathcal{I}$ the identity, is surjective from $D(\mathcal{B})$ into $\mathbb{H}$, i.e., the problem

$$\mathcal{B}\widetilde{V} + \widetilde{V} = F \quad \text{or, equivalently,} \quad \mathcal{A}\widetilde{U} + \mathcal{M}\widetilde{U} = F \tag{1.32}$$

admits a solution $\widetilde{V} \in D(\mathcal{B})$ ($\widetilde{U} \in D(\mathcal{A})$) for any $F \in \mathbb{H}$. Using the notation $F = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, F_5)$, the first equation of system $\mathcal{A}\widetilde{U} + \mathcal{M}\widetilde{U} = F$ reads

$$\widetilde{\mathbf{v}} = \widetilde{\mathbf{u}} - \mathbf{F}_1, \tag{1.33}$$

which can be substituted into the other equations to give

$$\begin{cases} -\mathrm{div}\,\mathbf{Ce}(\widetilde{\mathbf{u}}) + \mathrm{div}\,\mathbf{P}^T\mathbf{E} + \mathrm{div}\,\mathbf{R}^T\mathbf{H} + \mathrm{div}\,\boldsymbol{\beta}\widetilde{\theta} + \rho\widetilde{\mathbf{u}} = \mathbf{F}_2 + \rho\mathbf{F}_1, \\ \mathbf{Pe}(\widetilde{\mathbf{u}}) - \nabla \times \mathbf{H} + \mathbf{XE} + \boldsymbol{\alpha}\mathbf{H} + \mathbf{p}\widetilde{\theta} = \mathbf{F}_3 + \mathbf{Pe}(\mathbf{F}_1), \\ \mathbf{Re}(\widetilde{\mathbf{u}}) + \nabla \times \mathbf{E} + \boldsymbol{\alpha}\mathbf{E} + \mathbf{MH} + \mathbf{m}\widetilde{\theta} = \mathbf{F}_4 + \mathbf{Re}(\mathbf{F}_1), \\ \boldsymbol{\beta} : \mathbf{e}(\widetilde{\mathbf{u}}) - \dfrac{1}{T_0}\mathrm{div}\,\mathbf{K}\nabla\widetilde{\theta} + \mathbf{p} \cdot \mathbf{E} + \mathbf{m} \cdot \mathbf{H} + c_v\widetilde{\theta} = F_5 + \boldsymbol{\beta} : \mathbf{e}(\mathbf{F}_1). \end{cases} \tag{1.34}$$

Furthermore, $(1.34)_2$ allows us to express $\mathbf{E}$ in terms of $\mathbf{u}$, $\mathbf{H}$, θ and source terms $\mathbf{F}_3$ and $\mathbf{F}_1$; after substituting this expression in the remaining equations, resorting to the weak formulation of the obtained system and applying the Lax-Milgram lemma, we obtain a solution

$$(\tilde{\mathbf{u}}, \mathbf{H}, \tilde{\theta}) \in \mathbf{H}^1_{mD}(\Omega) \times \mathbf{H}_{gD}(\mathrm{curl}, \Omega) \times H^1_{tD}(\Omega)$$

for any $F \in \mathbb{H}$, satisfying the homogeneous boundary conditions appearing in the definition of $D(\mathcal{A})$. Of course, by (1.33), $\tilde{\mathbf{v}} \in \mathbf{H}^1_{mD}(\Omega)$ as well; also, one can verify that $\mathbf{E} \in \mathbf{H}_{eD}(\mathrm{curl}, \Omega)$ by taking the curl of the expression giving $\mathbf{E}$ in terms of the other unknowns and using equation $(1.34)_3$.

Step 6. By taking the divergence of $(1.24)_2$, considering the continuity equation (1.2) and integrating in time, we get[j]

$$\mathrm{div}\, \mathbf{D}(x,t) - \mathrm{div}\, \mathbf{D}(x,0) = \rho_e(x,t) - \rho_e(x,0), \quad x \in \Omega,\ t > 0.$$

Now, the point-wise version of hypothesis $(1.22)_1$ reads

$$\mathrm{div}\, \mathbf{D}(x,0) = \rho_e(x,0), \quad x \in \Omega,$$

hence we recover

$$\mathrm{div}\, \mathbf{D}(x,t) = \rho_e(x,t), \quad x \in \Omega,\ t > 0,$$

i.e., equation $(1.19)_2$. By $(1.22)_2$, a completely analogous argument yields

$$\mathrm{div}\, \mathbf{B}(x,t) = \mathrm{div}\, \mathbf{B}(x,0) = 0, \quad x \in \Omega,\ t > 0,$$

i.e., equation $(1.19)_3$. Therefore, equations $(1.19)_2$ and $(1.19)_3$ are not actually independent of $(1.19)_4$ and $(1.19)_5$, but they are a consequence of them and of (1.22).

□

Remark 1.2. The physical meaning of (1.22) is the following: the total volume electric charge must equal the total surface electric charge at $t = 0$, and hence at any $t > 0$; analogously, the total surface magnetic charge must vanish at $t = 0$, and hence at any $t > 0$.

1.5. *Energy Functional*

An energy evolution equation satisfied by the solution $(\mathbf{u}, \mathbf{E}, \mathbf{H}, \theta)$ of (1.24) can be formally derived. Indeed, upon defining

$$\begin{aligned}\mathcal{E}(t) \coloneqq &\frac{1}{2}\int_\Omega \rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \mathbf{Ce}(\mathbf{u}) : \mathbf{e}(\mathbf{u})\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \mathbf{XE}\cdot\mathbf{E}\,\mathrm{d}\Omega + \\ &+ \frac{1}{2}\int_\Omega \mathbf{MH}\cdot\mathbf{H}\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega c_v\theta^2\,\mathrm{d}\Omega + \int_\Omega \boldsymbol{\alpha}\mathbf{E}\cdot\mathbf{H}\,\mathrm{d}\Omega + \\ &+ \int_\Omega (\mathbf{p}\cdot\mathbf{E})\theta\,\mathrm{d}\Omega + \int_\Omega (\mathbf{m}\cdot\mathbf{H})\theta\,\mathrm{d}\Omega,\end{aligned}$$

[j]Here we make explicit the dependence on x and t for more clarity.

and

$$L(t) \coloneqq \int_{\Omega} \mathbf{f} \cdot \dot{\mathbf{u}} \, \mathrm{d}\Omega - \int_{\Omega} \mathbf{J} \cdot \mathbf{E} \, \mathrm{d}\Omega + \int_{\Omega} r\theta \, \mathrm{d}\Omega + \int_{\Gamma_{mN}} \mathbf{g} \cdot \dot{\mathbf{u}} \, \mathrm{d}\Gamma + \frac{1}{T_0} \int_{\Gamma_{tN}} \varrho\theta \, \mathrm{d}\Gamma, \quad (1.35)$$

then by taking the time derivative of $\mathcal{E}$, using equations (1.24) and integrating by parts, we get

$$\dot{\mathcal{E}}(t) + \frac{1}{T_0} \int_{\Omega} \mathbf{K}\nabla\theta \cdot \nabla\theta \, \mathrm{d}\Omega = L(t), \quad \forall t > 0. \quad (1.36)$$

The left-hand side of (1.36) can be thought of as the *actual internal power*, whereas the right-hand side as the *actual external power*. The word ‘actual’ (in place of *virtual*) is used here to stress the fact that (1.36) is *not* the variational formulation of system (1.24); indeed, it has been derived considering the *solution* of (1.24).

Remark 1.3. Note that the free electric charge density ρ_e, as well as boundary values b and d, do not appear in (1.35), since they just intervene in compatibility conditions (1.22).

2. Quasi-static Problem

2.1. *Nondimensionalization of the Equations*

With a view toward justifying the quasi-static hypothesis, i.e., the existence of two scalar fields φ and ζ – respectively, the electric and magnetic potentials – such that

$$\mathbf{E} = -\nabla\varphi \quad \text{and} \quad \mathbf{H} = -\nabla\zeta, \quad (2.1)$$

we nondimensionalize system (1.19) disregarding equations $(1.19)_2$ and $(1.19)_3$, which do not involve time derivatives of the unknowns; we shall come back to these equations later on. Let $V_j(x, \boldsymbol{\nu})$ denote the square root of the j-th eigenvalue of the acoustic tensor[k] associated with propagation direction $\boldsymbol{\nu}$ and evaluated at x, and let

$$\mathcal{L} \coloneqq \sup_{x,y\in\Omega} |x - y|, \quad V_+ \coloneqq \max_{j=1,2,3} \sup_{x\in\Omega} \sup_{|\boldsymbol{\nu}|=1} V_j(x, \boldsymbol{\nu}), \quad \mathcal{T} \coloneqq \frac{\mathcal{L}}{V_+}$$

be, respectively, the characteristic size of Ω, the maximum propagation speed for an elastic wave in Ω and the typical time for an elastic wave to travel along distance $\mathcal{L}$. Following Imperiale and Joly [18], we introduce[l]:

$$\begin{aligned}
&\rho_+ \coloneqq \sup_{x\in\Omega} \rho(x),\ P_+ \coloneqq \sup_{x\in\Omega} \sqrt{\|\mathbf{P}(x)\mathbf{P}(x)^T\|_2}, && R_+ \coloneqq \sup_{x\in\Omega} \sqrt{\|\mathbf{R}(x)\mathbf{R}(x)^T\|_2},\\
&\alpha_+ \coloneqq \sup_{x\in\Omega} \|\boldsymbol{\alpha}(x)\|_2, \quad p_+ \coloneqq \sup_{x\in\Omega} |\mathbf{p}(x)|, && m_+ \coloneqq \sup_{x\in\Omega} |\mathbf{m}(x)|,\\
&c_{v+} \coloneqq \sup_{x\in\Omega} c_v(x), \quad \beta_+ \coloneqq \sup_{x\in\Omega} \|\boldsymbol{\beta}(x)\|_2, && K_+ \coloneqq \sup_{x\in\Omega} \|\mathbf{K}(x)\|_2,
\end{aligned}$$

[k]We recall that the acoustic tensor $\mathbf{A}_{\boldsymbol{\nu}}$ associated with unit vector $\boldsymbol{\nu}$ (the propagation direction), a tensor field over Ω, is the second-order tensor defined by the following condition:

$$\mathbf{A}_{\boldsymbol{\nu}}\mathbf{a} \coloneqq \rho^{-1}\mathbf{C}\,[\mathbf{a} \otimes \boldsymbol{\nu}]\,\boldsymbol{\nu}, \quad \forall \mathbf{a} \in \mathbb{R}^3,$$

where ρ is the density and $\mathbf{C}$ the elasticity tensor.

[l]In the subsequent definitions, we use the notation $\|\mathbf{A}\|_2 \coloneqq \sqrt{\mathbf{A} : \mathbf{A}}$ for the norm of any second-order tensor $\mathbf{A}$.

and rewrite density and constitutive parameters as:

$$
\begin{array}{lll}
\rho(x) = \rho_+\, \rho_r\left(\frac{x}{\mathcal{L}}\right), & \mathbf{C}(x) = \rho_+ V_+^2\, \mathbf{C}_r\left(\frac{x}{\mathcal{L}}\right), & \mathbf{P}(x) = P_+\, \mathbf{P}_r\left(\frac{x}{\mathcal{L}}\right), \\
\mathbf{R}(x) = R_+ \mathbf{R}_r\left(\frac{x}{\mathcal{L}}\right), & \mathbf{X}(x) = \epsilon_0\, \mathbf{X}_r\left(\frac{x}{\mathcal{L}}\right), & \mathbf{M}(x) = \mu_0\, \mathbf{M}_r\left(\frac{x}{\mathcal{L}}\right), \\
\boldsymbol{\alpha}(x) = \alpha_+ \boldsymbol{\alpha}_r\left(\frac{x}{\mathcal{L}}\right), & \mathbf{p}(x) = p_+ \mathbf{p}_r\left(\frac{x}{\mathcal{L}}\right), & \mathbf{m}(x) = m_+ \mathbf{m}_r\left(\frac{x}{\mathcal{L}}\right), \\
c_v(x) = c_{v+} c_{vr}\left(\frac{x}{\mathcal{L}}\right), & \boldsymbol{\beta}(x) = \beta_+ \boldsymbol{\beta}_r\left(\frac{x}{\mathcal{L}}\right), & \mathbf{K}(x) = K_+ \mathbf{K}_r\left(\frac{x}{\mathcal{L}}\right).
\end{array}
$$

As for the unknowns, we write:

$$
\begin{array}{ll}
\mathbf{u}(x,t) = \mathcal{L}\, \mathbf{u}_r\left(\frac{x}{\mathcal{L}}, \frac{t}{\mathcal{T}}\right), & \mathbf{E}(x,t) = E^* \mathbf{E}_r\left(\frac{x}{\mathcal{L}}, \frac{t}{\mathcal{T}}\right), \\
\mathbf{H}(x,t) = H^* \mathbf{H}_r\left(\frac{x}{\mathcal{L}}, \frac{t}{\mathcal{T}}\right), & \theta(x,t) = T^* \theta_r\left(\frac{x}{\mathcal{L}}, \frac{t}{\mathcal{T}}\right),
\end{array}
$$

where ϵ_0 and μ_0 are, respectively, the electric permittivity and magnetic permeability of the vacuum, and E^*, H^* and T^* are to be properly chosen. We recall that the speed of light is given by $c_0 = (\epsilon_0 \mu_0)^{-\frac{1}{2}}$.

All in all, equations $(1.19)_1$ and $(1.19)_4$ to $(1.19)_6$ become (we still denote by x and t the scaled space and time variables):

$$
\left\{
\begin{aligned}
&\rho_r \ddot{\mathbf{u}}_r - \operatorname{div} \mathbf{C}_r \mathbf{e}(\mathbf{u}_r) + \left[\frac{P_+ E^*}{\rho_+ V_+^2}\right] \operatorname{div} \mathbf{P}_r^T \mathbf{E}_r + \\
&\qquad + \left[\frac{R_+ H^*}{\rho_+ V_+^2}\right] \operatorname{div} \mathbf{R}_r^T \mathbf{H}_r + \left[\frac{\beta_+ T^*}{\rho_+ V_+^2}\right] \operatorname{div} \boldsymbol{\beta}_r \theta_r = \mathbf{f}_r, \\
&\mathbf{X}_r \dot{\mathbf{E}}_r + \left[\frac{P_+}{\epsilon_0 E^*}\right] \mathbf{P}_r \mathbf{e}(\dot{\mathbf{u}}_r) + \left[\frac{\alpha_+ H^*}{\epsilon_0 E^*}\right] \boldsymbol{\alpha}_r \dot{\mathbf{H}}_r + \\
&\qquad + \left[\frac{p_+ T^*}{\epsilon_0 E^*}\right] \mathbf{p}_r \dot{\theta}_r - \left[\frac{\sqrt{\mu_0} H^*}{\sqrt{\epsilon_0} E^*}\right] \left[\frac{c_0}{V_+}\right] \nabla \times \mathbf{H}_r = -\mathbf{J}_r, \\
&\mathbf{M}_r \dot{\mathbf{H}}_r + \left[\frac{R_+}{\mu_0 H^*}\right] \mathbf{R}_r \mathbf{e}(\dot{\mathbf{u}}_r) + \left[\frac{\alpha_+ E^*}{\mu_0 H^*}\right] \boldsymbol{\alpha}_r \dot{\mathbf{E}}_r + \\
&\qquad + \left[\frac{m_+ T^*}{\mu_0 H^*}\right] \mathbf{m}_r \dot{\theta}_r + \left[\frac{\sqrt{\epsilon_0} E^*}{\sqrt{\mu_0} H^*}\right] \left[\frac{c_0}{V_+}\right] \nabla \times \mathbf{E}_r = \mathbf{0}, \\
&c_{vr} \dot{\theta}_r + \left[\frac{\beta_+}{c_{v+} T^*}\right] \boldsymbol{\beta}_r : \mathbf{e}(\dot{\mathbf{u}}_r) + \left[\frac{p_+ E^*}{c_{v+} T^*}\right] (\mathbf{p}_r \cdot \dot{\mathbf{E}}_r) + \\
&\qquad + \left[\frac{m_+ H^*}{c_{v+} T^*}\right] (\mathbf{m}_r \cdot \dot{\mathbf{H}}_r) - \left[\frac{K_+}{V_+ \mathcal{L} c_{v+} T_0}\right] \operatorname{div} \mathbf{K}_r \nabla \theta_r = r_r.
\end{aligned}
\right.
\tag{2.2}
$$

All equations hold for $x \in \widehat{\Omega}$ and $t > 0$, with $\widehat{\Omega} := \{x/\mathcal{L} : x \in \Omega\}$, and all coefficients between square parentheses are dimensionless. We now choose the units of measurement E^*, H^* and T^* in order that the following equalities hold:

$$
\sqrt{\epsilon_0} E^* = \sqrt{\mu_0} H^*, \quad \frac{P_+ E^*}{\rho_+ V_+^2} = \frac{P_+}{\epsilon_0 E^*}, \quad \frac{p_+ E^*}{c_{v+} T^*} = \frac{p_+ T^*}{\epsilon_0 E^*}.
$$

This yields

$$E^* = V_+\sqrt{\frac{\rho_+}{\epsilon_0}}, \quad H^* = V_+\sqrt{\frac{\rho_+}{\mu_0}}, \quad T^* = V_+\sqrt{\frac{\rho_+}{c_{v+}}}. \tag{2.3}$$

Note that this choice of E^*, H^* and T^* ensures symmetry of all other coupling coefficients as well, namely,

$$\frac{\beta_+ T^*}{\rho_+ V_+^2} = \frac{\beta_+}{c_{v+}T^*}, \quad \frac{R_+ H^*}{\rho_+ V_+^2} = \frac{R_+}{\mu_0 H^*}, \quad \frac{\alpha_+ H^*}{\epsilon_0 E^*} = \frac{\alpha_+ E^*}{\mu_0 H^*}, \quad \frac{m_+ T^*}{\mu_0 H^*} = \frac{m_+ H^*}{c_{v+} T^*}.$$

Upon setting

$$\delta := \frac{V_+}{c_0},$$

we can rewrite equations $(2.2)_2$ and $(2.2)_3$ respectively as follows:

$$\begin{aligned} \nabla \times \mathbf{E}_r &= -\delta \left(\mathbf{M}_r \dot{\mathbf{H}}_r + \kappa\, \mathbf{R}_r \mathbf{e}(\dot{\mathbf{u}}_r) + \alpha_+ c_0\, \boldsymbol{\alpha}_r \dot{\mathbf{E}}_r + \upsilon\, \mathbf{m}_r \dot{\theta}_r\right), \\ \nabla \times \mathbf{H}_r &= \delta \left(\mathbf{X}_r \dot{\mathbf{E}}_r + \chi\, \mathbf{P}_r \mathbf{e}(\dot{\mathbf{u}}_r) + \alpha_+ c_0\, \boldsymbol{\alpha}_r \dot{\mathbf{H}}_r + \varsigma\, \mathbf{p}_r \dot{\theta}_r + \mathbf{J}_r\right), \end{aligned} \tag{2.4}$$

with

$$\kappa := \frac{R_+}{V_+\sqrt{\mu_0 \rho_+}}, \quad \upsilon := \frac{m_+}{\sqrt{\mu_0 c_{v+}}}, \quad \chi := \frac{P_+}{V_+\sqrt{\epsilon_0 \rho_+}}, \quad \varsigma := \frac{p_+}{\sqrt{\epsilon_0 c_{v+}}}.$$

Considering the numerical values of density and elastic moduli for the most classic example of magneto-electro-thermo-elastic composite, i.e., $BaTiO_3$–$CoFe_2O_4$ with a volume fraction of $BaTiO_3$ equal to 0.6 (see Table 1 in the Appendix), it results

$$V_+ \simeq 5980\,\text{m/s}, \quad \rho_+ = 5600\,\text{kg/m}^3,$$

which give, along with the values of the other constitutive parameters,

$$\delta \simeq 2 \cdot 10^{-5}, \quad \alpha_+ c_0 \simeq 0.75, \quad \kappa \simeq 0.78, \quad \upsilon \simeq 0.3, \quad \chi \simeq 9, \quad \varsigma \simeq 2.3.$$

Therefore, if the right-hand sides of equations (2.4) remain bounded for any $t > 0$, then, in the limit $\delta \to 0$, we obtain

$$\nabla \times \mathbf{E}_r = \mathbf{0} \quad \text{and} \quad \nabla \times \mathbf{H}_r = \mathbf{0},$$

which means that there exist an electric potential φ_r and a magnetic potential ζ_r such that

$$\mathbf{E}_r = -\nabla\varphi_r \quad \text{and} \quad \mathbf{H}_r = -\nabla\zeta_r,$$

i.e., the quasi-static hypothesis (2.1).

2.2. *Quasi-static Problem*

As a result, we can remove equations $(1.19)_4$ and $(1.19)_5$ from system (1.19), introduce the new list of unknowns $\mathcal{U} := (\mathbf{u}, \varphi, \zeta, \theta)$ and rewrite (1.9)-(1.10) as

$$\begin{aligned} \boldsymbol{\sigma}(\mathcal{U}) &= \mathbf{Ce}(\mathbf{u}) + \mathbf{P}^T\nabla\varphi + \mathbf{R}^T\nabla\zeta - \boldsymbol{\beta}\theta, \\ \mathbf{D}(\mathcal{U}) &= \mathbf{Pe}(\mathbf{u}) - \mathbf{X}\nabla\varphi - \boldsymbol{\alpha}\nabla\zeta + \mathbf{p}\theta, \\ \mathbf{B}(\mathcal{U}) &= \mathbf{Re}(\mathbf{u}) - \boldsymbol{\alpha}\nabla\varphi - \mathbf{M}\nabla\zeta + \mathbf{m}\theta, \\ s(\mathcal{U}) &= \boldsymbol{\beta} : \mathbf{e}(\mathbf{u}) - \mathbf{p}\cdot\nabla\varphi - \mathbf{m}\cdot\nabla\zeta + c_v\theta, \\ \mathbf{q}(\theta) &= -\mathbf{K}\nabla\theta. \end{aligned} \tag{2.5}$$

Also, (1.19) becomes

$$\begin{cases} \rho\ddot{\mathbf{u}} - \operatorname{div}\boldsymbol{\sigma}(\mathcal{U}) = \mathbf{f} & x \in \Omega, \quad t > 0, \\ \operatorname{div}\mathbf{D}(\mathcal{U}) = \rho_e & x \in \Omega, \quad t > 0, \\ \operatorname{div}\mathbf{B}(\mathcal{U}) = 0 & x \in \Omega, \quad t > 0, \\ \dot{s}(\mathcal{U}) + \dfrac{1}{T_0}\operatorname{div}\mathbf{q}(\theta) = r & x \in \Omega, \quad t > 0, \end{cases} \tag{2.6}$$

with initial conditions

$$\begin{cases} \mathbf{u}(x,0) = \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega, \\ \dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}(0) = \mathbf{u}_1 \text{ in } \Omega, \\ \theta(x,0) = \theta(0) = \theta_0 \text{ in } \Omega, \end{cases} \tag{2.7}$$

and boundary conditions

$$\begin{cases} \boldsymbol{\sigma}(\mathcal{U})\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{mN}, & \mathbf{u} = \mathbf{0} \text{ on } \Gamma_{mD}, \\ \mathbf{D}(\mathcal{U})\cdot\mathbf{n} = d \text{ on } \Gamma_{eN}, & \varphi = 0 \text{ on } \Gamma_{eD}, \\ \mathbf{B}(\mathcal{U})\cdot\mathbf{n} = b \text{ on } \Gamma_{gN}, & \zeta = 0 \text{ on } \Gamma_{gD}, \\ -\mathbf{q}(\theta)\cdot\mathbf{n} = \varrho \text{ on } \Gamma_{tN}, & \theta = 0 \text{ on } \Gamma_{tD}. \end{cases} \tag{2.8}$$

Let us remark (see [25]) that there is no need to impose initial conditions on φ and ζ, since they are formally given by the unique solution (φ_0, ζ_0) of the following system, for given $(\mathbf{u}_0, \theta_0)$ [m]

$$\begin{cases} \operatorname{div}\mathbf{D}(\mathcal{U}(0)) = \operatorname{div}\left(\mathbf{Pe}(\mathbf{u}_0) - \mathbf{X}\nabla\varphi_0 - \boldsymbol{\alpha}\nabla\zeta_0 + \mathbf{p}\theta_0\right) = \rho_e(0) \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{B}(\mathcal{U}(0)) = \operatorname{div}\left(\mathbf{Re}(\mathbf{u}_0) - \boldsymbol{\alpha}\nabla\varphi_0 - \mathbf{M}\nabla\zeta_0 + \mathbf{m}\theta_0\right) = 0 \quad \text{in } \Omega, \\ \mathbf{D}(\mathcal{U}(0))\cdot\mathbf{n} = d(0) \text{ on } \Gamma_{eN},\ \varphi_0 = 0 \text{ on } \Gamma_{eD}, \\ \mathbf{B}(\mathcal{U}(0))\cdot\mathbf{n} = b(0) \text{ on } \Gamma_{gN},\ \zeta_0 = 0 \text{ on } \Gamma_{gD}. \end{cases}$$

[m]This statement will be detailed in Remark 2.3 and proven in Lemma 2.1.

2.3. ***Energy Functional***

As in the case of problem (1.24), we can define a quasi-static energy

$$\begin{aligned}\mathcal{E}_{qs}(t) &:= \frac{1}{2}\int_\Omega \rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \mathbf{Ce}(\mathbf{u}) : \mathbf{e}(\mathbf{u})\,\mathrm{d}\Omega + \\ &+ \frac{1}{2}\int_\Omega \mathbf{X}\nabla\varphi\cdot\nabla\varphi\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \mathbf{M}\nabla\zeta\cdot\nabla\zeta\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega c_v\theta^2\,\mathrm{d}\Omega + \\ &+ \int_\Omega \boldsymbol{\alpha}\nabla\varphi\cdot\nabla\zeta\,\mathrm{d}\Omega - \int_\Omega(\mathbf{p}\cdot\nabla\varphi)\theta\,\mathrm{d}\Omega - \int_\Omega(\mathbf{m}\cdot\nabla\zeta)\theta\,\mathrm{d}\Omega\end{aligned} \tag{2.9}$$

and a quasi-static external power

$$\begin{aligned}L_{qs}(t) &:= \int_\Omega \mathbf{f}\cdot\dot{\mathbf{u}}\,\mathrm{d}\Omega + \int_\Omega \dot{\rho}_e\varphi\,\mathrm{d}\Omega + \int_\Omega r\theta\,\mathrm{d}\Omega + \\ &+ \int_{\Gamma_{mN}} \mathbf{g}\cdot\dot{\mathbf{u}}\,\mathrm{d}\Gamma + \frac{1}{T_0}\int_{\Gamma_{tN}} \varrho\theta\,\mathrm{d}\Gamma - \int_{\Gamma_{eN}} \dot{d}\varphi\,\mathrm{d}\Gamma - \int_{\Gamma_{gN}} \dot{b}\zeta\,\mathrm{d}\Gamma\end{aligned} \tag{2.10}$$

to get the evolution equation (see [25])

$$\dot{\mathcal{E}}_{qs}(t) + \frac{1}{T_0}\int_\Omega \mathbf{K}\nabla\theta\cdot\nabla\theta\,\mathrm{d}\Omega = L_{qs}(t), \quad \forall t > 0. \tag{2.11}$$

Remark 2.1. Note that $\mathcal{E}(t) = \mathcal{E}_{qs}(t)$ for any $t \geqslant 0$, and that, unlike in the case of (1.36), here the time derivatives of ρ_e, d and b are present in the expression of the actual external power on the right-hand side of (2.11).

2.4. ***Existence and Uniqueness***

Set $T_0 = 1$ for simplicity and denote by $(\cdot,\cdot)$ the scalar product[n] in $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$. For any *fixed* $t \in (0,T)$, a weak version of (2.6)-(2.8) takes the following form:

$$A(\mathcal{U}(t),\mathcal{V}) = L(\mathcal{V}), \tag{2.12}$$

for all $\mathcal{V} = (\mathbf{v},\psi,\xi,\eta) \in \mathbf{H}^1_{mD}(\Omega)\times H^1_{eD}(\Omega)\times H^1_{gD}(\Omega)\times H^1_{tD}(\Omega)$, with

$$\mathcal{U}(t) \in \mathbf{H}^1_{mD}(\Omega)\times H^1_{eD}(\Omega)\times H^1_{gD}(\Omega)\times H^1_{tD}(\Omega),$$

where

$$\begin{aligned}A(\mathcal{U}(t),\mathcal{V}) &:= (\rho\ddot{\mathbf{u}}(t),\mathbf{v}) + c(\dot{\mathbf{u}}(t),\eta) + (c_v\dot{\theta}(t),\eta) - d(\dot{\varphi}(t),\eta) - e(\dot{\zeta}(t),\eta) + \\ &\quad + a_u(\mathbf{u}(t),\mathbf{v}) + b(\varphi(t),\mathbf{v}) - b(\psi,\mathbf{u}(t)) + f(\zeta(t),\mathbf{v}) - f(\xi,\mathbf{u}(t)) + \\ &\quad - c(\mathbf{v},\theta(t)) + a_\varphi(\varphi(t),\psi) + a_\zeta(\zeta(t),\xi) + g(\zeta(t),\psi) + g(\varphi(t),\xi) + \\ &\quad - d(\psi,\theta(t)) - e(\xi,\theta(t)) + a_\theta(\theta(t),\eta), \\ L(\mathcal{V}) &:= (\mathbf{f}(t),\mathbf{v}) + (r(t),\eta) + (\rho_e(t),\psi) + (\mathbf{g}(t),\mathbf{v})_{\mathbf{L}^2(\Gamma_{mN})} + \\ &\quad + (\varrho(t),\eta)_{L^2(\Gamma_{tN})} - (d(t),\psi)_{L^2(\Gamma_{eN})} - (b(t),\xi)_{L^2(\Gamma_{gN})}.\end{aligned} \tag{2.13}$$

[n]We shall denote by $\|\cdot\|$ the corresponding norm, unless noted otherwise. The scalar product in $L^2(\Gamma)$ or $\mathbf{L}^2(\Gamma)$, for $\Gamma \subset \partial\Omega$, will be denoted analogously, with a proper subscript.

where the bilinear forms $a_u(\cdot,\cdot)$, $a_\varphi(\cdot,\cdot)$, $a_\zeta(\cdot,\cdot)$, $a_\theta(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$, $d(\cdot,\cdot)$, $e(\cdot,\cdot)$, $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ are defined as follows:

$$\begin{aligned}
a_u(\mathbf{u},\mathbf{v}) &:= \int_\Omega \mathbf{Ce}(\mathbf{u}) : \mathbf{e}(\mathbf{v})\,\mathrm{d}\Omega, & a_\varphi(\varphi,\psi) &:= \int_\Omega \mathbf{X}\nabla\varphi\cdot\nabla\psi\,\mathrm{d}\Omega,\\
a_\zeta(\zeta,\xi) &:= \int_\Omega \mathbf{M}\nabla\zeta\cdot\nabla\xi\,\mathrm{d}\Omega, & a_\theta(\theta,\eta) &:= \int_\Omega \mathbf{K}\nabla\theta\cdot\nabla\eta\,\mathrm{d}\Omega,\\
b(\psi,\mathbf{u}) &:= \int_\Omega \mathbf{P}^T\nabla\psi : \mathbf{e}(\mathbf{u})\,\mathrm{d}\Omega, & c(\mathbf{u},\eta) &:= \int_\Omega \eta\boldsymbol{\beta} : \mathbf{e}(\mathbf{u})\,\mathrm{d}\Omega,\\
d(\varphi,\eta) &:= \int_\Omega \eta\mathbf{p}\cdot\nabla\varphi\,\mathrm{d}\Omega, & e(\zeta,\eta) &:= \int_\Omega \eta\mathbf{m}\cdot\nabla\zeta\,\mathrm{d}\Omega,\\
f(\xi,\mathbf{u}) &:= \int_\Omega \mathbf{R}^T\nabla\xi : \mathbf{e}(\mathbf{u})\,\mathrm{d}\Omega, & g(\psi,\zeta) &:= \int_\Omega \boldsymbol{\alpha}\nabla\psi\cdot\nabla\zeta\,\mathrm{d}\Omega.
\end{aligned} \tag{2.14}$$

Remark 2.2. Note that time derivatives of the unknowns are present in some terms in the definition of $A(\mathcal{U}(t),\mathcal{V})$. We shall give a precise meaning to such terms, as well as to initial conditions (2.7), in the statement of the existence and uniqueness theorem; furthermore we point out that all the terms containing first-order time derivatives are integrated against $\eta \in H^1_{tD}(\Omega)$.

Remark 2.3. As noted previously, initial conditions on φ and ζ are given by equations $(2.6)_2$ and $(2.6)_3$, along with the corresponding boundary conditions. Hence, $(\varphi_0,\zeta_0) \in H^1_{eD}(\Omega)\times H^1_{gD}(\Omega)$ is the unique solution to the following boundary value problem:

$$\begin{gathered}
a_\varphi(\varphi_0,\psi) + a_\zeta(\zeta_0,\xi) + g(\zeta_0,\psi) + g(\varphi_0,\xi) = (\rho_e(0),\psi) - (d(0),\psi)_{L^2(\Gamma_{eN})} +\\
-(b(0),\xi)_{L^2(\Gamma_{gN})} + b(\psi,\mathbf{u}_0) + f(\xi,\mathbf{u}_0) + d(\psi,\theta_0) + e(\xi,\theta_0),\\
\forall(\psi,\xi)\in H^1_{eD}(\Omega)\times H^1_{gD}(\Omega).
\end{gathered} \tag{2.15}$$

Lemma 2.1. *Problem* (2.15) *is well-posed.*

Proof. Hypothesis (1.17) implies that the submatrix

$$\begin{pmatrix} [\mathbf{X}] & [\boldsymbol{\alpha}] \\ [\boldsymbol{\alpha}] & [\mathbf{M}] \end{pmatrix}$$

of $[\mathbb{M}^c]$ is symmetric and positive definite. Let $W := H^1_{eD}(\Omega)\times H^1_{gD}(\Omega)$. Then, by definitions (2.14), the left-hand side of (2.15) is a continuous and coercive bilinear form (yet symmetric) over $W\times W$, and by the continuity of the trace operator, Poincaré's and Cauchy-Schwarz inequalities, the right-hand side of (2.15) is a bounded linear functional over W. The result follows by the Lax-Milgram theorem.

□

Before giving the statement of well-posedness for the quasi-static problem, we introduce the spaces $\mathbf{H}^*_{mD}(\Omega)$ and $H^*_{tD}(\Omega)$ as the *dual spaces* of, respectively, $\mathbf{H}^1_{mD}(\Omega)$ and $H^1_{tD}(\Omega)$, i.e., the spaces of continuous linear functionals over $\mathbf{H}^1_{mD}(\Omega)$ and $H^1_{tD}(\Omega)$. As $\mathbf{H}^1_{mD}(\Omega)$

and $H^1_{tD}(\Omega)$ are dense and continuously embedded in $\mathbf{L}^2(\Omega)$ and $L^2(\Omega)$ respectively, then we have [6, 28]

$$\mathbf{H}^1_{mD}(\Omega) \subset \mathbf{L}^2(\Omega) \simeq \left[\mathbf{L}^2(\Omega)\right]^* \subset \mathbf{H}^*_{mD}(\Omega),$$
$$H^1_{tD}(\Omega) \subset L^2(\Omega) \simeq \left[L^2(\Omega)\right]^* \subset H^*_{tD}(\Omega),$$

where we have used an analogous notation for the dual spaces of $\mathbf{L}^2(\Omega)$ and $L^2(\Omega)$, and the identification is realized by means of the Riesz-Fréchet representation theorem, being $\mathbf{L}^2(\Omega)$ and $L^2(\Omega)$ the *pivot spaces*. In the sequel, we adopt an analogous notation: the linear actions of $\boldsymbol{\ell} \in \mathbf{H}^*_{mD}(\Omega)$ or $\ell \in H^*_{tD}(\Omega)$ on the corresponding elements $\mathbf{v} \in \mathbf{H}^1_{mD}(\Omega)$ and $\eta \in H^1_{tD}(\Omega)$ are denoted by $\langle \boldsymbol{\ell}, \mathbf{v}\rangle$ and $\langle \ell, \eta\rangle$, without including subscripts, as meaning will be clear from the context[o].

The results about existence and uniqueness for the quasi-static problem are summarized in the following statement.

Theorem 2.1. *Let $T > 0$. Assume the following regularity properties on the initial data:*

$$(\mathbf{u}_0, \mathbf{u}_1, \theta_0) \in \mathbf{H}^1_{mD}(\Omega) \times \mathbf{L}^2(\Omega) \times H^1_{tD}(\Omega),$$

the following regularity properties on source and boundary values:

$$\begin{cases} \mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega)), \\ \rho_e \in H^1(0,T;L^2(\Omega)) \cap C^0([0,T];L^2(\Omega)), \\ r \in L^2(0,T;L^2(\Omega)), \\ \mathbf{g} \in H^2(0,T;\mathbf{L}^2(\Gamma_{mN})) \cap C^1([0,T];\mathbf{L}^2(\Gamma_{mN})), \\ d \in H^1(0,T;L^2(\Gamma_{eN})) \cap C^0([0,T];L^2(\Gamma_{eN})), \\ b \in H^1(0,T;L^2(\Gamma_{gN})) \cap C^0([0,T];L^2(\Gamma_{gN})), \\ \varrho \in L^2(0,T;L^2(\Gamma_{tN})), \end{cases}$$

and the following compatibility conditions:

$$\begin{cases} \mathbf{g}(0) = \boldsymbol{\sigma}(\mathbf{u}_0, \varphi_0, \zeta_0, \theta_0)\mathbf{n} & \text{on } \Gamma_{mN}, \\ d(0) = \mathbf{D}(\mathbf{u}_0, \varphi_0, \zeta_0, \theta_0)\cdot \mathbf{n} & \text{on } \Gamma_{eN}, \\ b(0) = \mathbf{B}(\mathbf{u}_0, \varphi_0, \zeta_0, \theta_0)\cdot \mathbf{n} & \text{on } \Gamma_{gN}, \\ \varrho(0) = -\mathbf{q}(\theta_0)\cdot \mathbf{n} & \text{on } \Gamma_{tN}. \end{cases}$$

Then, problem (2.12) *admits a unique solution $\mathcal{U} = (\mathbf{u}, \varphi, \zeta, \theta)$ such that:*

$$\begin{cases} \mathbf{u} \in L^2(0,T;\mathbf{H}^1_{mD}(\Omega)) \cap C^0([0,T];\mathbf{L}^2(\Omega)), \\ \dot{\mathbf{u}} \in L^2(0,T;\mathbf{L}^2(\Omega)), \\ \rho\ddot{\mathbf{u}} \in L^2(0,T;\mathbf{H}^*_{mD}(\Omega)), \\ \varphi \in L^2(0,T;H^1_{eD}(\Omega)), \\ \zeta \in L^2(0,T;H^1_{gD}(\Omega)), \\ \theta \in L^2(0,T;H^1_{tD}(\Omega)), \\ c_v\dot{\theta} + \boldsymbol{\beta} : \mathbf{e}(\dot{\mathbf{u}}) - \mathbf{p}\cdot\nabla\dot{\varphi} - \mathbf{m}\cdot\nabla\dot{\zeta} \in L^2(0,T;H^*_{tD}(\Omega)). \end{cases}$$

[o]Clearly, if $\boldsymbol{\ell} \in \mathbf{L}^2(\Omega)$ then $\langle \boldsymbol{\ell}, \mathbf{v}\rangle = (\boldsymbol{\ell}, \mathbf{v})$, $\forall \mathbf{v} \in \mathbf{H}^1_{mD}(\Omega)$ and analogously for $\ell \in L^2(\Omega)$ and $\eta \in H^1_{tD}(\Omega)$.

Remark 2.4. Based on the solution regularity, the left-hand side terms $(\rho\ddot{\mathbf{u}}(t), \mathbf{v})$ and $(c_v\dot{\theta}(t), \eta) + c(\dot{\mathbf{u}}(t), \eta) - d(\dot{\varphi}(t), \eta) - e(\dot{\zeta}(t), \eta)$ in (2.12) are actually to be thought of as duality pairings.

Proof. We split the proof into six steps.

Step 1. First we consider, as in Step 2 of the proof of Theorem 1.1, the following auxiliary problem for any $t \geqslant 0$:

$$\begin{cases} \operatorname{div} \mathbf{Ce}(\widehat{\mathbf{u}}) = \mathbf{0}, & x \in \Omega, \\ \mathbf{Ce}(\widehat{\mathbf{u}})\mathbf{n} = \mathbf{g} & \text{on } \Gamma_{mN}, \\ \widehat{\mathbf{u}} = \mathbf{0} & \text{on } \Gamma_{mD}. \end{cases} \tag{2.16}$$

Analogously to (1.25)-(1.26), provided

$$\mathbf{g} \in H^2(0, T; \mathbf{L}^2(\Gamma_{mN})) \cap C^1([0, T]; \mathbf{L}^2(\Gamma_{mN})),$$

this problem admits a unique solution

$$\widehat{\mathbf{u}} \in H^2(0, T; \mathbf{H}^1_{mD}(\Omega)) \cap C^1([0, T]; \mathbf{H}^1_{mD}(\Omega)). \tag{2.17}$$

Let $\overline{\mathbf{u}} := \mathbf{u} - \widehat{\mathbf{u}}$; we rewrite (2.12) in terms of the list of unknowns $\overline{\mathcal{U}} := (\overline{\mathbf{u}}, \varphi, \zeta, \theta)$. Then, by (2.13), we have

$$A(\overline{\mathcal{U}}(t), \mathcal{V}) = \overline{L}(\mathcal{V}), \tag{2.18}$$

with[p]

$$\begin{aligned} \overline{L}(\mathcal{V}) := {} & L(\mathcal{V}) - (\mathbf{g}(t), \mathbf{v})_{\mathbf{L}^2(\Gamma_{mN})} - (\rho\, \partial_{tt}\widehat{\mathbf{u}}(t), \mathbf{v}) + \\ & - (\boldsymbol{\beta} : \mathbf{e}(\widehat{\mathbf{u}}(t)), \eta) + (\mathbf{Pe}(\widehat{\mathbf{u}}(t)), \nabla\psi) + (\mathbf{Re}(\widehat{\mathbf{u}}(t)), \nabla\xi) \\ = {} & (\mathbf{f}(t), \mathbf{v}) - (\rho\, \partial_{tt}\widehat{\mathbf{u}}(t), \mathbf{v}) + (\rho_e(t), \psi) + (r(t), \eta) - (\boldsymbol{\beta} : \mathbf{e}(\widehat{\mathbf{u}}(t)), \eta) + \\ & + (\mathbf{Pe}(\widehat{\mathbf{u}}(t)), \nabla\psi) + (\mathbf{Re}(\widehat{\mathbf{u}}(t)), \nabla\xi) + \\ & + (\varrho(t), \eta)_{L^2(\Gamma_{tN})} - (d(t), \psi)_{L^2(\Gamma_{eN})} - (b(t), \xi)_{L^2(\Gamma_{gN})}. \end{aligned}$$

Step 2. Let $\{\mathbf{v}_k\}_{k=1}^{\infty}$ and $\{\eta_k\}_{k=1}^{\infty}$ be orthonormal bases for spaces $\mathbf{H}^1_{mD}(\Omega)$ and $H^1_{tD}(\Omega)$, respectively. Fix $p \in \mathbb{N}$ and define $\mathbf{V}_p(\Omega, \Gamma_{mD}) := \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, $V_p(\Omega, \Gamma_{tD}) := \operatorname{span}\{\eta_1, \ldots, \eta_p\}$. These spaces are such that their union for all $p \in \mathbb{N}$ is mutually dense in $\mathbf{H}^1_{mD}(\Omega)$ and $H^1_{tD}(\Omega)$, respectively. Define

$$\overline{\mathbf{u}}_p(t) := \sum_{k=1}^{p} u_k(t)\mathbf{v}_k, \quad \theta_p(t) := \sum_{k=1}^{p} \vartheta_k(t)\eta_k. \tag{2.19}$$

Furthermore, let $(\varphi_p(t), \zeta_p(t))$ be the unique solution to the following problem:

Given $(\overline{\mathbf{u}}_p(t), \theta_p(t)) \in \mathbf{H}^1_{mD}(\Omega) \times H^1_{tD}(\Omega)$,
find $(\varphi_p(t), \zeta_p(t)) \in H^1_{eD}(\Omega) \times H^1_{gD}(\Omega)$ *such that*

[p]Note that, by (2.17), $\rho\, \partial_{tt}\widehat{\mathbf{u}} \in L^2(0, T; \mathbf{L}^2(\Omega))$.

$$
\begin{aligned}
&a_\varphi(\varphi_p(t),\psi) + a_\zeta(\zeta_p(t),\xi) + g(\zeta_p(t),\psi) + g(\varphi_p(t),\xi) = \\
&= (\rho_e(t),\psi) - (d(t),\psi)_{L^2(\Gamma_{eN})} - (b(t),\xi)_{L^2(\Gamma_{gN})} + b(\psi,\overline{\mathbf{u}}_p(t)) + f(\xi,\overline{\mathbf{u}}_p(t)) + \\
&+ d(\psi,\theta_p(t)) + e(\xi,\theta_p(t)) + (\mathbf{Pe}(\widehat{\mathbf{u}}(t)),\nabla\psi) + (\mathbf{Re}(\widehat{\mathbf{u}}(t)),\nabla\xi), \\
&\forall(\psi,\xi) \in H^1_{eD}(\Omega) \times H^1_{gD}(\Omega).
\end{aligned}
\tag{2.20}
$$

By (2.18), the sets of coefficients $\{u_k(t)\}_{k=1}^p$ and $\{\vartheta_k(t)\}_{k=1}^p$ are determined by the solution of the following system of *ordinary differential equations*:

$$
\begin{cases}
\langle \rho\, \partial_{tt}\overline{\mathbf{u}}_p(t), \mathbf{v}_k\rangle + a_u(\overline{\mathbf{u}}_p(t),\mathbf{v}_k) + b(\varphi_p(t),\mathbf{v}_k) + f(\zeta_p(t),\mathbf{v}_k) - c(\mathbf{v}_k,\theta_p(t)) = \\
= (\mathbf{f}(t),\mathbf{v}_k) - (\rho\,\partial_{tt}\widehat{\mathbf{u}}(t),\mathbf{v}_k), \quad \forall k \in \{1,\ldots,p\}, \\
\langle c_v\dot{\theta}_p(t),\eta_k\rangle + c(\partial_t\overline{\mathbf{u}}_p(t),\eta_k) - d(\dot{\varphi}_p(t),\eta_k) - e(\dot{\zeta}_p(t),\eta_k) + a_\theta(\theta_p(t),\eta_k) = \\
= (r(t),\eta_k) - (\boldsymbol{\beta} : \mathbf{e}(\widehat{\mathbf{u}}(t)),\eta_k) + (\varrho(t),\eta_k)_{L^2(\Gamma_{tN})}, \quad \forall k \in \{1,\ldots,p\},
\end{cases}
\tag{2.21}
$$

with initial conditions such that the following strong convergences hold:

$$
\begin{cases}
\overline{\mathbf{u}}_p(0) = \overline{\mathbf{u}}_{p,0} \to \overline{\mathbf{u}}_0 & \text{in } \mathbf{H}^1_{mD}(\Omega), \\
\partial_t\overline{\mathbf{u}}_p(0) = \overline{\mathbf{u}}_{p,1} \to \overline{\mathbf{u}}_1 & \text{in } \mathbf{L}^2(\Omega), \\
\theta_p(0) = \theta_{p,0} \to \theta_0 & \text{in } H^1_{tD}(\Omega),
\end{cases}
\tag{2.22}
$$

where $\overline{\mathbf{u}}_0 := \overline{\mathbf{u}}(0) = \mathbf{u}(0) - \widehat{\mathbf{u}}(0) = \mathbf{u}_0 - \widehat{\mathbf{u}}(0)$, $\overline{\mathbf{u}}_1 := \partial_t\overline{\mathbf{u}}(0) = \dot{\mathbf{u}}(0) - \partial_t\widehat{\mathbf{u}}(0) = \mathbf{u}_1 - \partial_t\widehat{\mathbf{u}}(0)$. The linear ordinary differential equations theory guarantees (see [31]) that there exists $t_p > 0$ such that system (2.21)-(2.22) admits a local solution in time interval $[0, t_p]$.

Step 3. We rewrite the energy functional associated with the problem in terms of the bilinear forms introduced above:

$$
2\overline{\mathcal{E}}(t) = (\rho\,\partial_t\overline{\mathbf{u}}(t), \partial_t\overline{\mathbf{u}}(t)) + a_u(\overline{\mathbf{u}}(t),\overline{\mathbf{u}}(t)) + Q_{\mathbb{M}^c}(\varphi(t),\zeta(t),\theta(t)),
$$

where

$$
\begin{aligned}
&Q_{\mathbb{M}^c} : H^1_{eD}(\Omega) \times H^1_{gD}(\Omega) \times L^2(\Omega) \to \mathbb{R}, \\
&Q_{\mathbb{M}^c}(\psi,\xi,\eta) = a_\varphi(\psi,\psi) + a_\zeta(\xi,\xi) + (c_v\eta,\eta) + 2g(\psi,\xi) - 2d(\psi,\eta) - 2e(\xi,\eta)
\end{aligned}
\tag{2.23}
$$

is the quadratic form associated with the coupling matrix. Note that by hypothesis (1.17), there exist a coercivity constant $\mathscr{M} > 0$ and a continuity constant $\mathfrak{M} > 0$ (depending on the lowest and greatest eigenvalues of $\mathbb{M}^c$, respectively) such that

$$
\mathscr{M}\left(\|\nabla\psi\|^2 + \|\nabla\xi\|^2 + \|\eta\|^2\right) \leqslant Q_{\mathbb{M}^c}(\psi,\xi,\eta) \leqslant \mathfrak{M}\left(\|\nabla\psi\|^2 + \|\nabla\xi\|^2 + \|\eta\|^2\right), \tag{2.24}
$$

for all $(\psi,\xi,\eta) \in H^1_{eD}(\Omega) \times H^1_{gD}(\Omega) \times L^2(\Omega)$.

One can prove that an evolution equation analogous to (2.11) is satisfied by the energy $\overline{\mathcal{E}}_p$ associated with $(\overline{\mathbf{u}}_p, \varphi_p, \zeta_p, \theta_p)$, namely,

$$
\partial_t\overline{\mathcal{E}}_p(t) + a_\theta(\theta_p(t),\theta_p(t)) = \overline{L}_p(t). \tag{2.25}
$$

Indeed, first multiply $(2.21)_1$ by $u_k'(t)$, $(2.21)_2$ by $\vartheta_k'(t)$, then make the sum on k ranging from 1 to p; finally, take the time derivative of (2.20) and replace ψ by $\varphi_p(t)$. By adding the three equations thus obtained, we get (2.25) with

$$2\overline{\mathcal{E}}_p(t) := (\rho\,\partial_t\overline{\mathbf{u}}_p(t), \partial_t\overline{\mathbf{u}}_p(t)) + a_u(\overline{\mathbf{u}}_p(t), \overline{\mathbf{u}}_p(t)) + Q_{\mathbb{M}^c}(\varphi_p(t), \zeta_p(t), \theta_p(t)),$$

$$\begin{aligned}\overline{L}_p(t) := {} & (\mathbf{f}(t), \partial_t\overline{\mathbf{u}}_p(t)) - (\rho\,\partial_{tt}\widehat{\mathbf{u}}(t), \partial_t\overline{\mathbf{u}}_p(t)) + (\dot{\rho}_e(t), \varphi_p(t)) + (r(t), \theta_p(t)) + \\ & -(\boldsymbol{\beta} : \mathbf{e}(\widehat{\mathbf{u}}(t)), \theta_p(t)) + (\mathbf{Pe}(\widehat{\mathbf{u}}(t)), \nabla\varphi_p(t)) + (\mathbf{Re}(\widehat{\mathbf{u}}(t)), \nabla\zeta_p(t)) + \\ & -(\dot{d}(t), \varphi_p(t))_{L^2(\Gamma_{eN})} - (\dot{b}(t), \zeta_p(t))_{L^2(\Gamma_{gN})} + (\varrho(t), \theta_p(t))_{L^2(\Gamma_{mN})}.\end{aligned}$$

Step 4. We shall prove now that the solution $\overline{\mathcal{E}}_p$ of (2.25) is bounded for any $t \in [0, t_p]$ by a constant depending on T, which allows us to extend the time interval to $[0, T]$. First, we look for a bound on the external power $\overline{L}_p(t)$ depending on $\overline{\mathcal{E}}_p(t)$. We use the continuity of the trace operator, Poincaré's and Young's inequalities to infer the existence of constants $c_{r,1}$, $c_{r,2}$ and $c_{r,3}$ such that:

$$\begin{aligned}\overline{L}_p(t) \leqslant {} & C_0(t) + \frac{1}{2}\Big(\|\partial_t\overline{\mathbf{u}}_p(t)\|^2 + (c_{r,1}^2 + 1)\|\nabla\varphi_p(t)\|^2 + \\ & +(c_{r,2}^2 + 1)\|\nabla\zeta_p(t)\|^2 + \|\theta_p(t)\|^2\Big) + \frac{\delta_0}{2}c_{r,3}^2\|\nabla\theta_p(t)\|^2,\end{aligned}$$

$$\begin{aligned}2C_0(t) := {} & \|\mathbf{f}(t)\|^2 + \|\rho\,\partial_{tt}\widehat{\mathbf{u}}(t)\|^2 + \|\dot{\rho}_e(t)\|^2 + \|\boldsymbol{\beta} : \mathbf{e}(\widehat{\mathbf{u}}(t))\|^2 + \|\mathbf{Pe}(\widehat{\mathbf{u}}(t))\|^2 + \\ & +\|\mathbf{Re}(\widehat{\mathbf{u}}(t))\|^2 + \|r(t)\|^2 + \|\dot{d}(t)\|^2_{L^2(\Gamma_{eN})} + \|\dot{b}(t)\|^2_{L^2(\Gamma_{gN})} + \frac{1}{\delta_0}\|\varrho(t)\|^2_{L^2(\Gamma_{tN})}.\end{aligned}$$

Let $K > 0$ denote the coercivity constant of $a_\theta(\cdot,\cdot)$. We select δ_0 such that $\widetilde{K} := K - \frac{\delta_0}{2}c_{r,3}^2 > 0$. By (2.24) we can determine a constant $C > 0$ such that

$$\begin{aligned}& (c_{r,1}^2 + 1)\|\nabla\varphi_p(t)\|^2 + (c_{r,2}^2 + 1)\|\nabla\zeta_p(t)\|^2 + \|\theta_p(t)\|^2 \leqslant \\ & \leqslant C\,Q_{\mathbb{M}^c}(\varphi_p(t), \zeta_p(t), \theta_p(t)) \leqslant 2C\,\overline{\mathcal{E}}_p(t).\end{aligned}$$

Hence, there exist $C_0(t)$ and C_1 such that

$$\partial_t\overline{\mathcal{E}}_p(t) + \widetilde{K}\|\nabla\theta_p(t)\|^2 \leqslant L_p(t) \leqslant C_0(t) + C_1\overline{\mathcal{E}}_p(t),\ \ \forall t \in [0, t_p],$$

which gives, upon integrating in time,

$$\overline{\mathcal{E}}_p(t) + \widetilde{K}\int_0^t \|\nabla\theta_p(s)\|^2 \mathrm{d}s \leqslant \overline{\mathcal{E}}_p(0) + \int_0^t (C_0(s) + C_1\overline{\mathcal{E}}_p(s))\mathrm{d}s,$$

for any $t \in [0, t_p]$. Now, note that $\overline{\mathcal{E}}_p(0)$ contains the terms $(\rho\,\partial_t\overline{\mathbf{u}}_p(0), \partial_t\overline{\mathbf{u}}_p(0))$, $a_u(\overline{\mathbf{u}}_p(0), \overline{\mathbf{u}}_p(0))$, $(c_v\theta_p(0), \theta_p(0))$, which are bounded by (2.22). To bound the remaining terms, we have to find bounds on $\|\nabla\varphi_p(0)\|$ and $\|\nabla\zeta_p(0)\|$. This can be accomplished by taking $t = 0$ in (2.20), replacing ψ and ξ by, respectively, $\varphi_p(0)$ and $\zeta_p(0)$, and then using hypothesis (1.17), Cauchy-Schwarz and Young's inequality, together with the continuity of the trace operator; all in all (we omit details), one finds that there exist a constant $c > 0$ depending on the material parameters such that

$$\begin{aligned}\|\nabla\varphi_p(0)\|^2 + \|\nabla\zeta_p(0)\|^2 \leqslant {} & c\,\Big(\|\rho_e(0)\|^2 + \|d(0)\|^2_{L^2(\Gamma_{eN})} + \|b(0)\|^2_{L^2(\Gamma_{gN})} + \\ & +\|\mathbf{Pe}(\widehat{\mathbf{u}}(0))\|^2 + \|\mathbf{Re}(\widehat{\mathbf{u}}(0))\|^2 + 2(\|\theta_p(0)\|^2 + \|\nabla\overline{\mathbf{u}}_p(0)\|^2)\Big),\end{aligned}$$

and since the right-hand side is bounded by (2.22), $\|\nabla\varphi_p(0)\|$ and $\|\nabla\zeta_p(0)\|$ are bounded. Hence, $\overline{\mathcal{E}}_p(0)$ is bounded. Moreover, as $C_0(t)$ depends only on the data, we can integrate up to $T > 0$ and get

$$\int_0^t C_0(s)\mathrm{d}s \leqslant \int_0^T C_0(s)\mathrm{d}s = \overline{C}_0,$$

$$\begin{aligned}2\overline{C}_0 \coloneqq\ & \|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\dot{\rho}_e\|^2_{L^2(0,T;L^2(\Omega))} + \|r\|^2_{L^2(0,T;L^2(\Omega))} + \|\rho\,\partial_{tt}\widehat{\mathbf{u}}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \\ & + \|\boldsymbol{\beta} : \mathbf{e}(\widehat{\mathbf{u}})\|^2_{L^2(0,T;L^2(\Omega))} + \|\mathbf{Pe}(\widehat{\mathbf{u}})\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{Re}(\widehat{\mathbf{u}})\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \\ & + \|\dot{d}\|^2_{L^2(0,T;L^2(\Gamma_{eN}))} + \|\dot{b}\|^2_{L^2(0,T;L^2(\Gamma_{gN}))} + \frac{1}{\delta_0}\|\varrho\|^2_{L^2(0,T;L^2(\Gamma_{tN}))}.\end{aligned}$$

All in all, we finally obtain the existence of positive constants $\widetilde{K}$, k_1 and k_2 such that

$$\overline{\mathcal{E}}_p(t) + \widetilde{K}\int_0^t \|\nabla\theta_p(s)\|^2\mathrm{d}s \leqslant k_1 + k_2\int_0^t \overline{\mathcal{E}}_p(s)\mathrm{d}s, \quad \forall t \in [0, t_p],$$

whence, by Gronwall's lemma, for any $t \in [0, t_p]$ it results

$$\overline{\mathcal{E}}_p(t) \leqslant k_1 e^{k_2 t} \leqslant k_1 e^{k_2 T} < +\infty, \quad \forall\, T \in [t_p, +\infty). \tag{2.26}$$

Hence, as the solution of (2.25) is uniformly bounded in $[0, t_p]$, we can extend the maximal existence interval to $[0, T]$. As a consequence, we also get

$$\widetilde{K}\int_0^T \|\nabla\theta_p(s)\|^2\mathrm{d}s \leqslant k_1 e^{k_2 T}. \tag{2.27}$$

Step 5. From the last two bounds, we infer the weak convergence of subsequences still denoted $(\overline{\mathbf{u}}_p, \varphi_p, \zeta_p, \theta_p)$ as follows:

$$\begin{cases} \overline{\mathbf{u}}_p \rightharpoonup \overline{\mathbf{u}} & \text{in } L^2(0,T;\mathbf{H}^1_{mD}(\Omega)), \\ \partial_t\overline{\mathbf{u}}_p \rightharpoonup \partial_t\overline{\mathbf{u}} & \text{in } L^2(0,T;\mathbf{L}^2(\Omega)), \\ \varphi_p \rightharpoonup \varphi & \text{in } L^2(0,T;H^1_{eD}(\Omega)), \\ \zeta_p \rightharpoonup \zeta & \text{in } L^2(0,T;H^1_{gD}(\Omega)), \\ \theta_p \rightharpoonup \theta & \text{in } L^2(0,T;H^1_{tD}(\Omega)). \end{cases} \tag{2.28}$$

Moreover, to get an estimate on $\rho\,\partial_{tt}\overline{\mathbf{u}}_p$, we select $\mathbf{v} \in \mathbf{H}^1_{mD}(\Omega)$ such that $\|\mathbf{v}\|_{\mathbf{H}^1_{mD}(\Omega)} \leqslant 1$; the projection theorem for separable Hilbert spaces allows us to write $\mathbf{v} = \mathbf{v}^1 + \mathbf{v}^2$, with $\mathbf{v}^1 \in \operatorname{span}\{\mathbf{v}_k\}_{k=1}^p$ and $(\mathbf{v}^2, \mathbf{v}_k)_{\mathbf{H}^1_{mD}(\Omega)} = 0$ for $k = 1, \dots, p$. Then, $(2.21)_1$ implies that

$$\begin{aligned}\langle\rho\,\partial_{tt}\overline{\mathbf{u}}_p(t), \mathbf{v}\rangle = \langle\rho\,\partial_{tt}\overline{\mathbf{u}}_p(t), \mathbf{v}^1\rangle = & - a_u(\overline{\mathbf{u}}_p(t), \mathbf{v}^1) - b(\varphi_p(t), \mathbf{v}^1) - f(\zeta_p(t), \mathbf{v}^1) + \\ & + c(\mathbf{v}^1, \theta_p(t)) + (\mathbf{f}(t), \mathbf{v}^1) - (\rho\,\partial_{tt}\widehat{\mathbf{u}}(t), \mathbf{v}^1),\end{aligned}$$

whence, by using Cauchy-Schwarz and Young's inequalities, integrating in time and taking into account the boundedness of the energy $\mathcal{E}_p(t)$, we infer the existence of positive constants k_3 and k_4 depending on the material parameters such that

$$\|\rho\,\partial_{tt}\overline{\mathbf{u}}_p\|_{L^2(0,T;\mathbf{H}^*_{mD}(\Omega))} \leqslant k_1 k_3 T e^{k_2 T} + k_4\left(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\rho\,\partial_{tt}\widehat{\mathbf{u}}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}\right),$$

therefore $\rho\,\partial_{tt}\overline{\mathbf{u}}_p$ is bounded in $L^2(0,T;\mathbf{H}^*_{mD}(\Omega))$ and hence it admits a subsequence still denoted $\rho\,\partial_{tt}\overline{\mathbf{u}}_p$ such that

$$\rho\,\partial_{tt}\overline{\mathbf{u}}_p \rightharpoonup \rho\,\partial_{tt}\overline{\mathbf{u}} \quad \text{in} \quad L^2(0,T;\mathbf{H}^*_{mD}(\Omega)). \tag{2.29}$$

Now, let $\Xi_p(t) \coloneqq c_v\dot{\theta}_p(t) + \boldsymbol{\beta} : \partial_t\mathbf{e}(\overline{\mathbf{u}}(t)) - \mathbf{p}\cdot\nabla\dot{\varphi}(t) - \mathbf{m}\cdot\nabla\dot{\zeta}(t)$; with an analogous procedure, if we take $\eta \in H^1_{tD}(\Omega)$ with $\|\eta\|_{H^1_{tD}(\Omega)} \leqslant 1$ and write $\eta = \eta^1 + \eta^2$, $\eta^1 \in \operatorname{span}\{\eta_1,\dots,\eta_k\}$ and $(\eta^2,\eta_k)_{H^1_{tD}(\Omega)} = 0$ for $k = 1,\dots,p$, then $(2.21)_2$ reads

$$\begin{gathered}\langle\Xi_p(t),\eta\rangle = \langle\Xi_p(t),\eta^1\rangle = \\ = -a_\theta(\theta_p(t),\eta^1) + (r(t),\eta^1) + (\boldsymbol{\beta} : \mathbf{e}(\widehat{\mathbf{u}}(t)),\eta^1) + (\varrho(t),\eta^1)_{L^2(\Gamma_{tN})},\end{gathered}$$

whence, by using Cauchy-Schwarz and Young's inequalities, together with the continuity of the trace operator, integrating in time and taking into account (2.27), we can determine two positive constants k_5 and k_6 depending on the material parameters such that

$$\begin{aligned}\|\Xi_p\|_{L^2(0,T;H^*_{tD}(\Omega))} \leqslant{}& k_5\|\nabla\theta_p\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + k_6\Big(\|r\|^2_{L^2(0,T;L^2(\Omega))} + \\ &+ \|\boldsymbol{\beta} : \mathbf{e}(\widehat{\mathbf{u}})\|^2_{L^2(0,T;L^2(\Omega))} + \|\varrho\|^2_{L^2(0,T;L^2(\Gamma_{mN}))}\Big)\,;\end{aligned}$$

Ξ_p is thus bounded and admits a subsequence still denoted Ξ_p such that

$$\Xi_p \rightharpoonup \Xi \quad \text{in} \quad L^2(0,T;H^*_{tD}(\Omega)). \tag{2.30}$$

Now, spaces $\mathbf{V}_p(\Omega,\Gamma_{mD})$ and $V_p(\Omega,\Gamma_{tD})$ are dense, respectively, in $\mathbf{H}^1_{mD}(\Omega)$ and $H^1_{tD}(\Omega)$. By multiplying (2.20) and the equations of system (2.21) by a test function $\lambda \in \mathcal{D}(0,T)$, integrating in time, passing to the limit, taking into account (2.28)-(2.29)-(2.30) and exploiting the arbitrariness of λ, we obtain

$$\begin{gathered}A(\overline{\mathcal{U}}(t),\mathcal{V}) = \overline{L}(\mathcal{V}), \\ \forall\mathcal{V} \in \mathbf{H}^1_{mD}(\Omega)\times H^1_{eD}(\Omega)\times H^1_{gD}(\Omega)\times H^1_{tD}(\Omega),\ t\in(0,T).\end{gathered}$$

Again, from (2.28)-(2.29)-(2.30), we infer the regularity properties of the solution:

$$\begin{cases}\overline{\mathbf{u}} \in L^2(0,T;\mathbf{H}^1_{mD}(\Omega)) \cap C^0([0,T];\mathbf{L}^2(\Omega)), \\ \partial_t\overline{\mathbf{u}} \in L^2(0,T;\mathbf{L}^2(\Omega)), \\ \rho\,\partial_{tt}\overline{\mathbf{u}} \in L^2(0,T;\mathbf{H}^*_{mD}(\Omega)) \\ \varphi \in L^2(0,T;H^1_{eD}(\Omega)), \\ \zeta \in L^2(0,T;H^1_{gD}(\Omega)), \\ \theta \in L^2(0,T;H^1_{tD}(\Omega)), \\ c_v\dot{\theta} + \boldsymbol{\beta} : \partial_t\mathbf{e}(\overline{\mathbf{u}}) - \mathbf{p}\cdot\nabla\dot{\varphi} - \mathbf{m}\cdot\nabla\dot{\zeta} \in L^2(0,T;H^*_{tD}(\Omega))\end{cases}$$

It remains to show that the weak limits $\overline{\mathbf{u}}$, $\partial_t\overline{\mathbf{u}}$ and θ satisfy the imposed initial conditions $\overline{\mathbf{u}}(0) = \overline{\mathbf{u}}_0$, $\partial_t\overline{\mathbf{u}}(0) = \overline{\mathbf{u}}_1$ and $\theta(0) = \theta_0$. As in [25], to prove that $\overline{\mathbf{u}}(0) = \overline{\mathbf{u}}_0$, we choose $\lambda \in C^1([0,T])$ such that $\lambda(0) = 1$ and $\lambda(T) = 0$; an application of (2.28) yields

$$\int_0^T (\partial_t\overline{\mathbf{u}}_p(t),\mathbf{v})\lambda(t)\,\mathrm{d}t \to \int_0^T (\partial_t\overline{\mathbf{u}}(t),\mathbf{v})\lambda(t)\,\mathrm{d}t, \quad \forall\mathbf{v}\in\mathbf{H}^1_{mD}(\Omega);$$

upon integrating by parts, we get

$$-(\overline{\mathbf{u}}_p(0),\mathbf{v}) - \int_0^T (\overline{\mathbf{u}}_p(t),\mathbf{v})\lambda'(t)\,\mathrm{d}t \to -(\overline{\mathbf{u}}(0),\mathbf{v}) - \int_0^T (\overline{\mathbf{u}}(t),\mathbf{v})\lambda'(t)\,\mathrm{d}t.$$

Now, since $\int_0^T (\overline{\mathbf{u}}_p(t),\mathbf{v})\lambda'(t)\,\mathrm{d}t \to \int_0^T (\overline{\mathbf{u}}(t),\mathbf{v})\lambda'(t)\,\mathrm{d}t$ by (2.28), we have

$$(\overline{\mathbf{u}}_p(0),\mathbf{v}) \to (\overline{\mathbf{u}}(0),\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}^1_{mD}(\Omega),$$

i.e., $\overline{\mathbf{u}}_p(0) \rightharpoonup \overline{\mathbf{u}}(0)$ in $\mathbf{L}^2(\Omega)$. By $(2.22)_1$ and the uniqueness of limit we get $\overline{\mathbf{u}}(0) = \overline{\mathbf{u}}_0$. Analogously we show that $\partial_t\overline{\mathbf{u}}(0) = \overline{\mathbf{u}}_1$ and $\theta(0) = \theta_0$.

Step 6. The solution $\overline{\mathcal{U}}(t)$ is unique. By contradiction, if there exist two solutions $(\overline{\mathbf{u}}^*, \varphi^*, \zeta^*, \theta^*)$ and $(\overline{\mathbf{u}}_*, \varphi_*, \zeta_*, \theta_*)$ of (2.18), set $\mathbf{w} := \overline{\mathbf{u}}^* - \overline{\mathbf{u}}_*$, $\psi := \varphi^* - \varphi_*$, $\xi := \zeta^* - \zeta_*$, $\eta := \theta^* - \theta_*$, then $\mathbf{W} := (\mathbf{w}, \psi, \xi, \eta)$, and consider the problem

$$\begin{cases} \rho\ddot{\mathbf{w}} - \operatorname{div}\boldsymbol{\sigma}(\mathbf{W}) = \mathbf{0}, \\ \operatorname{div}\mathbf{D}(\mathbf{W}) = 0, \\ \operatorname{div}\mathbf{B}(\mathbf{W}) = 0, \\ \dot{s}(\mathbf{W}) + \operatorname{div}\mathbf{q}(\eta) = 0, \end{cases}$$

with vanishing initial and boundary conditions. Denote by $\mathcal{E}(\mathbf{W}(t))$ the energy associated with $\mathbf{W}$ at time t; then, by (2.11) we get

$$\dot{\mathcal{E}}(\mathbf{W}(t)) \leqslant \dot{\mathcal{E}}(\mathbf{W}(t)) + a_\theta(\eta(t),\eta(t)) = 0,$$

whence, by integration, $\mathcal{E}(\mathbf{W}(t)) \leqslant \mathcal{E}(\mathbf{W}(0)) = 0$. This implies in turn that $a_\theta(\eta(t),\eta(t)) = 0$ and thus $\mathbf{W} = \mathbf{0}$. □

3. Another justification of the quasi-static assumption

In this subsection, we outline a study of the limit as $\delta \to 0$ of the solution to problem (1.19)-(1.20)-(1.21). We start by rewriting the nondimensionalized system (2.2):

$$\begin{cases} \rho_r\ddot{\mathbf{u}}_r - \operatorname{div}\mathbf{C}_r\mathbf{e}(\mathbf{u}_r) + \chi\operatorname{div}\mathbf{P}_r^T\mathbf{E}_r + \kappa\operatorname{div}\mathbf{R}_r^T\mathbf{H}_r + \gamma\operatorname{div}\boldsymbol{\beta}_r\theta_r = \mathbf{f}_r, \\ \mathbf{X}_r\dot{\mathbf{E}}_r + \chi\,\mathbf{P}_r\mathbf{e}(\dot{\mathbf{u}}_r) + \alpha_+c_0\,\boldsymbol{\alpha}_r\dot{\mathbf{H}}_r + \varsigma\,\mathbf{p}_r\dot{\theta}_r - \delta^{-1}\,\nabla\times\mathbf{H}_r = -\mathbf{J}_r, \\ \mathbf{M}_r\dot{\mathbf{H}}_r + \kappa\,\mathbf{R}_r\mathbf{e}(\dot{\mathbf{u}}_r) + \alpha_+c_0\,\boldsymbol{\alpha}_r\dot{\mathbf{E}}_r + \upsilon\,\mathbf{m}_r\dot{\theta}_r + \delta^{-1}\,\nabla\times\mathbf{E}_r = \mathbf{0}, \\ c_{vr}\dot{\theta}_r + \gamma\,\boldsymbol{\beta}_r : \mathbf{e}(\dot{\mathbf{u}}_r) + \varsigma\,(\mathbf{p}_r\cdot\dot{\mathbf{E}}_r) + \upsilon\,(\mathbf{m}_r\cdot\dot{\mathbf{H}}_r) - \lambda\operatorname{div}\mathbf{K}_r\nabla\theta_r = r_r, \end{cases} \tag{3.1}$$

where we have set[q]

$$\gamma := \frac{\beta_+}{V_+\sqrt{\rho_+c_{v+}}} \simeq 30, \quad \lambda := \frac{K_+}{V_+\mathcal{L}c_{v+}T_0} \simeq 1.6\cdot 10^{-7}.$$

[q]To calculate λ, we take $T_0 = 297$ K as a reference temperature, as in [20]; for what concerns the thermal conductivity, we take the weighted average of the values provided in [27], since numerical values for the $BaTiO_3$–$CoFe_2O_4$ composite with volume fraction 0.6 of $BaTiO_3$ are not available in literature. As for the length scale, we take $\mathcal{L} = 3$ cm.

We recall that all the equations hold for $x \in \widehat{\Omega}$ (we use an analogous notation for boundary partitions) and $t > 0$, and that x and t are dimensionless. The boundary conditions are

$$\begin{cases} \boldsymbol{\sigma}_r(\widetilde{\mathcal{U}}_r)\mathbf{n}_r = \mathbf{g}_r & \text{on } \widehat{\Gamma}_{mN}, \\ -\mathbf{q}_r(\theta_r)\cdot\mathbf{n}_r = \varrho_r & \text{on } \widehat{\Gamma}_{tN}, \end{cases} \qquad \begin{cases} \mathbf{u}_r = \mathbf{0} & \text{on } \widehat{\Gamma}_{mD}, \\ \mathbf{E}_r \times \mathbf{n}_r = \mathbf{0} & \text{on } \widehat{\Gamma}_{eD}, \\ \mathbf{H}_r \times \mathbf{n}_r = \mathbf{0} & \text{on } \widehat{\Gamma}_{gD}, \\ \theta_r = 0 & \text{on } \widehat{\Gamma}_{tD}, \end{cases} \tag{3.2}$$

where $\widetilde{\mathcal{U}}_r \coloneqq (\mathbf{u}_r, \mathbf{E}_r, \mathbf{H}_r, \theta_r)$, $\mathbf{n}_r$ is the outer unit normal vector field over $\partial\widehat{\Omega}$, $\boldsymbol{\sigma}_r(\widetilde{\mathcal{U}}_r) \coloneqq \mathbf{C}_r\mathbf{e}(\mathbf{u}_r) - \kappa\mathbf{R}_r^T\mathbf{H}_r - \chi\mathbf{P}_r^T\mathbf{E}_r - \gamma\boldsymbol{\beta}_r\theta_r$ and $\mathbf{q}_r(\theta_r) \coloneqq -\mathbf{K}_r\nabla\theta_r$. Again, we omit boundary conditions on the normal components of the magnetic induction $\mathbf{B}_r$ and of the electric displacement $\mathbf{D}_r$ as they just impose compatilibility conditions on the choice of initial values $\mathbf{u}_r^0$, $\mathbf{E}_r^0$, $\mathbf{H}_r^0$ and θ_r^0 and of the initial nondimensional electric charge density $\rho_{e_r}^0$.

A first hint that the limit solution of (3.1) as $\delta \to 0$ is such that $\nabla \times \mathbf{E}_r = \mathbf{0}$ and $\nabla \times \mathbf{H}_r = \mathbf{0}$ can be obtained by expressing the L^2–norm of $\nabla \times \mathbf{E}_r^0$ and of $\nabla \times \mathbf{H}_r^0$ in terms of the L^2–norm of $\nabla \times \mathbf{E}_0$ and of $\nabla \times \mathbf{H}_0$. Indeed, for any $x \in \Omega$, we have

$$\mathbf{E}_r^0\left(\frac{x}{\mathcal{L}}\right) = \frac{1}{E^*}\mathbf{E}_0(x) = \frac{1}{V_+}\sqrt{\frac{\epsilon_0}{\rho_+}}\mathbf{E}_0(x) = \frac{\delta}{V_+^2\sqrt{\mu_0\rho_+}}\mathbf{E}_0(x),$$

whence

$$\nabla \times \mathbf{E}_r^0\left(\frac{x}{\mathcal{L}}\right) = \frac{\delta\mathcal{L}}{V_+^2\sqrt{\mu_0\rho_+}}\nabla \times \mathbf{E}_0(x).$$

Upon integrating the squared euclidean norm of both members of this equality, we find

$$\int_\Omega \left|\nabla \times \mathbf{E}_r^0\left(\frac{x}{\mathcal{L}}\right)\right|^2 \mathrm{d}\Omega = \mathcal{L}^3 \int_{\widehat{\Omega}} |\nabla \times \mathbf{E}_r^0(y)|^2\,\mathrm{d}\widehat{\Omega} =$$

$$= \mathcal{L}^3\|\nabla \times \mathbf{E}_r^0\|^2_{\mathbf{L}^2(\widehat{\Omega})} = \frac{\delta^2\mathcal{L}^2}{V_+^4\mu_0\rho_+}\|\nabla \times \mathbf{E}_0\|^2_{\mathbf{L}^2(\Omega)},$$

that is to say,

$$\|\nabla \times \mathbf{E}_r^0\|_{\mathbf{L}^2(\widehat{\Omega})} = \frac{\delta}{V_+^2\sqrt{\mu_0\rho_+\mathcal{L}}}\|\nabla \times \mathbf{E}_0\|_{\mathbf{L}^2(\Omega)};$$

analogously, we obtain

$$\|\nabla \times \mathbf{H}_r^0\|_{\mathbf{L}^2(\widehat{\Omega})} = \frac{\delta}{V_+^2\sqrt{\epsilon_0\rho_+\mathcal{L}}}\|\nabla \times \mathbf{H}_0\|_{\mathbf{L}^2(\Omega)}.$$

The denominators appearing on the right-hand sides of these equalities are, of course, not dimensionless, and have the following numerical values:

$$V_+^2\sqrt{\mu_0\rho_+\mathcal{L}} \simeq 5.2 \cdot 10^5\,\mathrm{V\,m^{-1/2}} = O(\delta^{-1})\,\mathrm{V\,m^{-1/2}},$$

$$V_+^2\sqrt{\epsilon_0\rho_+\mathcal{L}} \simeq 1.38 \cdot 10^3\,\mathrm{A\,m^{-1/2}} = O(\delta^{-3/5})\,\mathrm{A\,m^{-1/2}},$$

therefore, unless $\|\nabla \times \mathbf{E}_0\|_{\mathbf{L}^2(\Omega)} = O(\delta^{-2})\,\mathrm{V\,m^{-1/2}}$ and $\|\nabla \times \mathbf{H}_0\|_{\mathbf{L}^2(\Omega)} = O(\delta^{-8/5})\,\mathrm{A\,m^{-1/2}}$, which is never the case in applications, we have

$$\lim_{\delta\to 0}\|\nabla \times \mathbf{E}_r^0\|_{\mathbf{L}^2(\widehat{\Omega})} = \lim_{\delta\to 0}\|\nabla \times \mathbf{H}_r^0\|_{\mathbf{L}^2(\widehat{\Omega})} = 0.$$

Conclusions

We presented a mathematical model to describe linear magneto-electro-thermo-elastic materials, as well as the proofs of the well-posedness for the problem formulated in its most general setting (dynamic) and for the quasi-static problem, based on the assumption that the electric and magnetic fields can be expressed as gradients of the corresponding potentials. We set forth a first validation of this hypothesis by carrying out a formal nondimensionalization procedure on the equations of the dynamic problem. A rigorous mathematical justification analogous to that set forth in [18] for a piezoelectric material needs further work, since the dynamic problem has been solved in the context of the Hille-Yosida theory, obtaining a smooth solution in time, whereas the solution of the quasi-static problem has been obtained in a weak form by the Faedo-Galerkin method.

A problem that has already been addressed [5] is the deduction of a *plate* model for a magneto-electro-thermo-elastic sensor/actuator, based on the quasi-static assumption, by means of the asymptotic expansion method, taking into account another *small* parameter ε tending to zero and characterizing the ratio of the plate thickness to the plate diameter. Another interesting extension for what concerns applications is the study of a laminated structure (plate-like or shell-like) including a thin magneto-electro-thermo-elastic layer; for the case of a piezoelectric layer see, e.g., [29] and [30].

Finally, in view of applications, a natural problem to be dealt with at a later stage is to come up with efficient numerical methods to perform simulations of the equations involved in the problem.

Appendix

Numerical Values of the Materials Constants

The table below lists numerical values of the material constants for a $BaTiO_3$–$CoFe_2O_4$ composite with 0.6 volume fraction of barium titanate. We took Table 1 in [22] as a reference. The two diagonal components M_{11} and M_{22} of the magnetic permeability tensor assume *negative* values in [22]. This would contradict our hypothesis (1.12), but it is actually a widespread error in literature, based on the incorrect determination of the sign of these coefficients for the pure cobalt ferrite $CoFe_2O_4$, as pointed out in [14], [20], [21] and [32]. Therefore, we corrected the values of these two components by changing their sign. In our notation, $\boldsymbol{\beta}$ denotes the thermal stress tensor, whereas in [22] it denotes the thermal *expansion* tensor. For $\boldsymbol{\gamma}$ the thermal expansion tensor and $\mathbf{C}$ the elasticity tensor, the thermal stress tensor $\boldsymbol{\beta}$ is given by the relation $\boldsymbol{\beta} = \mathbf{C}\boldsymbol{\gamma}$, which we used to infer the values of the thermal stress components. Also, since no constitutive assumption concerning entropy, as well as heat influx, is made in [22], numerical values for the calorific capacity and the thermal conductivities are unavailable in [22]. However, an estimate of these values can be retrieved, respectively, from [20] and [27].

Table 1. Material properties of a magneto-electro-thermo-elastic $BaTiO_3$–$CoFe_2O_4$ composite with 0.6 volume fraction of $BaTiO_3$.

Elastic moduli		**Magnetic permeabilities**	
$C_{1111} = C_{2222}$ (GPa)	200	$M_{11} = M_{22}$ (10^{-4} N s^2/C^2)	1.5
C_{1122} (GPa)	110	M_{33} (10^{-4} N s^2/C^2)	0.75
$C_{1133} = C_{2233}$ (GPa)	110	**Piezomagnetic constants**	
C_{3333} (GPa)	190	$R_{311} = R_{322}$ (N/A m)	200
$C_{2323} = C_{3131}$ (GPa)	45	R_{333} (N/A m)	260
C_{1212} (GPa)	45	R_{113} (N/A m)	180
Piezoelectric constants		**Magnetoelectric constants**	
$P_{311} = P_{322}$ (C/m^2)	-3.5	$\alpha_{11} = \alpha_{22}$ (10^{-12} N s/V C)	6
P_{333} (C/m^2)	11	α_{33} (10^{-12} N s/V C)	2500
Dielectric permittivities		**Pyroelectric constant**	
$X_{11} = X_{22}$ (10^{-9} C^2/N m^2)	0.9	p_3 (10^{-5} C/m^2 K)	-12.4
X_{33} (10^{-9} C^2/N m^2)	7.5	**Pyromagnetic constant**	
Thermal stresses		m_3 (10^{-3} N/A m K)	5.92
$\beta_{11} = \beta_{22}$ (10^6 N/K m^2)	4.86	**Density**	
β_{33} (10^6 N/K m^2)	4.32	ρ (kg/m^3)	5600
Thermal conductivities		**Calorific capacity**	
K_{33} (W/m K)	2.85	c_v (J/m^3 K^2)	325

The Hille-Yosida Theorem

The statement of the Hille-Yosida theorem for a linear non-homogeneous differential equation employed in the paper is the following (see [6]).

Theorem 3.1. *Let* $\mathbb{H}$ *be a Hilbert space,* $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ *a linear operator with domain* $D(\mathcal{A})$*, a linear subspace of* $\mathbb{H}$*. Let* $\mathcal{A}$ *be a* maximal monotone *operator. Then, for any* $U_0 \in D(\mathcal{A})$ *and* $F \in C^1([0,T];\mathbb{H})$*, the problem*

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + \mathcal{A}U = F, \quad t > 0, \\ U(0) = U_0 \end{cases}$$

admits a unique strong solution

$$U \in C^1([0,T];\mathbb{H}) \cap C^0([0,T];D(\mathcal{A})).$$

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Chapitre 4

Capteurs et Actionneurs en forme de Plaque

Nous formulons maintenant le problème quasi-statique sur un domaine en forme de plaque, d'épaisseur $h^\varepsilon = \varepsilon h$ étant h une longueur de référence et ε un petit paramètre qui tendra vers zéro, et utilisons la méthode des développements asymptotiques en puissances d'ε afin d'en déduire un modèle de plaque bidimensionnel. Dans ce qui suit on considère quatre types de conditions aux limites différents pour ce qui concerne les inconnues électromagnétiques, chaque ensemble de conditions aux limites visant à modéliser un comportement de capteur et/ou actionneur de type piézoélectrique et/ou piézomagnétique. L'analyse asymptotique requiert un choix précis des mises à l'échelle concernant les potentiels électrique et magnétique, suivant les conditions aux limites caractérisant le problème de départ [63]. Dans tous les quatre problèmes limites, le champs de déplacement est de type Kirchhoff-Love, la variation de température est indépendante de la coordonnée d'épaisseur, et chaque problème est découplé – un aspect typique de la théorie des plaques – en un problème de flexion et un problème membranaire totalement ou partiellement couplé ; la nature de ce dernier couplage est liée aux conditions aux limites du problème de départ. En particulier, l'équation d'évolution qui caractérise le problème de flexion tient compte d'un effet d'inertie de rotation. Nous montrons, de plus, que les solutions des quatre problèmes limites bidimensionnels ainsi obtenus sont les limites faibles des correspondantes solutions des problèmes quasi-statiques de départ lorsque $\varepsilon \to 0$.

Nous présentons ci-après la version étendue d'un article publié dans la revue *Mathematics and Mechanics of Solids*, écrit en collaboration avec G. Geymonat, F. Krasucki et M. Serpilli.

4.1 An Asymptotic Plate Model for Magneto-Electro-Thermo-Elastic Sensors and Actuators

In this paper, we consider a linear model of magneto-electro-thermo-elastic plates, behaving either as piezoelectric sensors or piezomagnetic actuators, based on the quasi-static assumption on the electric and magnetic fields, whereby both fields can be expressed as gradients of the corresponding potentials. This assumption was justified by means of a nondimensionalization of the equations governing the problem in its general setting in [9], wherein a proof of well-posedness for this problem, along with its quasi-static counterpart, was also accomplished.

The behavior of the plate-like body under study, as to whether it represents a sensor or an actuator, of piezoelectric or piezomagnetic nature, is determined by four different sets of boundary conditions [64]. Based on the three-dimensional formulations of the four corresponding problems, we apply the asymptotic expansion method as the thickness of the plate approaches zero, in the case of a homogeneous anisotropic material. Accordingly, we obtain four different two-dimensional plate models: the sensor-actuator model (referring to a plate behaving as a piezoelectric sensor and a piezomagnetic actuator), the actuator-sensor model (referring to a plate behaving as a piezoelectric actuator and a piezomagnetic sensor), the actuator model (according to which the plate behaves as a piezoelectric and piezomagnetic actuator) and the sensor model (according to which the plate behaves as a piezoelectric and piezomagnetic sensor). We validate the asymptotic procedure carried out in each of the four cases by showing weak convergence results. The four two-dimensional plate problems are obtained, as in [63], with different scaling assumptions on the electric and magnetic potentials. On the other hand, they all present common features: for one, the displacement field is always of Kirchhoff-Love type; for two, the temperature variation field is always independent of the thickness coordinate; for three, each problem decouples into a flexural problem – governing the evolution of the transversal displacement of the plate and taking account of an inertia effect involving the mean curvature of the deformed middle surface – and a certain partially or totally coupled membrane problem. In the sensor-actuator model, the membrane problem involves in-plane displacement, temperature variation and electric potential: it is therefore a *thermo-piezoelectric* problem, the applied magnetic potential playing the role of source term. In the actuator-sensor case, the roles of the two potentials are exchanged with respect to the sensor-actuator case, and thus we find a *thermo-piezomagnetic* membrane problem, the applied electric potential being part of source terms [1]. In the actuator case, the membrane problem is *thermo-elastic*, as it just involves in-plane displacement and temperature variation, both the applied electric and magnetic potentials playing the role of source terms. Finally, in the sensor case, the membrane problem is completely coupled, meaning that it involves in-plane displacement, temperature variation, electric potential and magnetic potential; it is then a *magneto-electro-thermo-elastic* problem. Numerical values of the reduced coefficients can be explicitly computed in each of the four cases; as an example, we reported in Table 4.1 of Appendix 2 the values of such coefficients in the actuator case.

Notation

Throughout the paper, $\omega \subset \mathbb{R}^2$ denotes a smooth domain in the plane spanned by vectors $\mathbf{e}_1 \equiv (1, 0)$ and $\mathbf{e}_2 \equiv (0, 1)$, with boundary γ; $\gamma_0 \subset \gamma$ is a measurable subset of γ, with strictly positive length measure; $\gamma_1 \coloneqq \gamma \backslash \gamma_0$ is the complement of γ_0 with respect to γ; finally, $0 < \varepsilon < 1$ is a dimensionless *small* real parameter which shall

1. We shall not treat this case in detail for the sake of brevity.

tend to zero. For each ε, we define

$$\Omega^\varepsilon := \omega \times (-h^\varepsilon, h^\varepsilon), \qquad \Gamma^\varepsilon := \gamma \times (-h^\varepsilon, h^\varepsilon)$$
$$\Gamma_0^\varepsilon := \gamma_0 \times (-h^\varepsilon, h^\varepsilon), \qquad \Gamma_\pm^\varepsilon := \omega \times \{\pm h^\varepsilon\},$$

with $h^\varepsilon > 0$. Hence the boundary of Ω^ε is partitioned into the lateral face Γ^ε and the upper and lower faces Γ_+^ε and Γ_-^ε; the lateral face is itself partitioned as $\Gamma^\varepsilon = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon$, with $\Gamma_1^\varepsilon := \gamma_1 \times (-h^\varepsilon, h^\varepsilon)$. Moreover, we let $\widehat{\Gamma}^\varepsilon := \Gamma_\pm^\varepsilon \cup \Gamma_1^\varepsilon = \partial\Omega^\varepsilon \backslash \Gamma_0^\varepsilon$, the complement of Γ_0^ε with respect to $\partial\Omega^\varepsilon$. For notational convenience, at times we left tacit the time-dependence of a field Φ. For the sake of brevity, a one-parameter family of fields $\{\Phi(\varepsilon)\}_{\varepsilon>0}$ is referred to as *sequence*. Scalars are denoted by light-face letters, vector and tensor fields of any order by bold-face letters. Let $\mathbf{H}^1(\Omega^\varepsilon) := [H^1(\Omega^\varepsilon)]^3$; for $\Xi^\varepsilon \subset \partial\Omega^\varepsilon$, we define

$$H^1(\Omega^\varepsilon, \Xi^\varepsilon) := \left\{ v^\varepsilon \in H^1(\Omega^\varepsilon); \;\; v^\varepsilon = 0 \text{ on } \Xi^\varepsilon \right\},$$
$$\mathbf{H}^1(\Omega^\varepsilon, \Xi^\varepsilon) := \left\{ \mathbf{v}^\varepsilon = (v_i^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon); \;\; \mathbf{v}^\varepsilon = \mathbf{0} \text{ on } \Xi^\varepsilon \right\},$$

and denote their respective duals as $H^*(\Omega^\varepsilon, \Xi^\varepsilon)$ and $\mathbf{H}^*(\Omega^\varepsilon, \Xi^\varepsilon)$. For $\mathbf{v}$ a vector field, $|\mathbf{v}|$ denotes the euclidean norm of $\mathbf{v}$; for Φ a scalar or vector field, $|\Phi|_{0,\Omega^\varepsilon}$ and $\|\Phi\|_{1,\Omega^\varepsilon}$ denote, respectively, the $L^2(\Omega^\varepsilon)$-norm and the $H^1(\Omega^\varepsilon)$-norm of Φ (analogous notations are used for the $L^2(\Xi^\varepsilon)$-norm of Φ, with Ξ^ε a subset of $\partial\Omega^\varepsilon$). We also employ Einstein's usual summation convention; Greek indices take the values 1 and 2, Roman indices range from 1 to 3.

4.1.1 Constitutive Laws

In magneto-electro-thermo-elastic materials the mechanical, electric, magnetic and thermal behaviors are coupled. In the case of the quasi-static approximation for Maxwell's equations, the electric field $\mathbf{E}^\varepsilon$ and the magnetic field $\mathbf{H}^\varepsilon$ can be expressed through two potential functions, i.e., $E_i^\varepsilon := -\partial_i^\varepsilon \varphi^\varepsilon$ and $H_i^\varepsilon := -\partial_i^\varepsilon \zeta^\varepsilon$, where φ^ε and ζ^ε denote, respectively, the electric potential and the magnetic potential. Thus, the magneto-electro-thermo-elastic state is defined by the quadruplet $\mathcal{U}^\varepsilon := (\mathbf{u}^\varepsilon, \varphi^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon)$ where $\mathbf{u}^\varepsilon = (u_i^\varepsilon)$ and θ^ε represent, respectively, the displacement field and the variation of temperature. The interaction between these four different behaviors is described by the following set of constitutive laws:

$$\begin{cases} \sigma_{ij}^\varepsilon(\mathcal{U}^\varepsilon) = C_{ijk\ell} e_{k\ell}^\varepsilon(\mathbf{u}^\varepsilon) + P_{kij}\partial_k^\varepsilon \varphi^\varepsilon + R_{kij}\partial_k^\varepsilon \zeta^\varepsilon - \beta_{ij}\theta^\varepsilon, \\ D_i^\varepsilon(\mathcal{U}^\varepsilon) = P_{ik\ell} e_{k\ell}^\varepsilon(\mathbf{u}^\varepsilon) - X_{ij}\partial_j^\varepsilon \varphi^\varepsilon - \alpha_{ij}\partial_j^\varepsilon \zeta^\varepsilon + p_i\theta^\varepsilon, \\ B_i^\varepsilon(\mathcal{U}^\varepsilon) = R_{ik\ell} e_{k\ell}^\varepsilon(\mathbf{u}^\varepsilon) - \alpha_{ij}\partial_j^\varepsilon \varphi^\varepsilon - M_{ij}\partial_j^\varepsilon \zeta^\varepsilon + m_i\theta^\varepsilon, \\ \mathcal{S}^\varepsilon(\mathcal{U}^\varepsilon) = \beta_{ij} e_{ij}^\varepsilon(\mathbf{u}^\varepsilon) - p_i\partial_i^\varepsilon \varphi^\varepsilon - m_i\partial_i^\varepsilon \zeta^\varepsilon + c_v\theta^\varepsilon, \\ q_i^\varepsilon(\theta^\varepsilon) = -K_{ij}\partial_j^\varepsilon \theta^\varepsilon, \end{cases}$$

where $\boldsymbol{\sigma}^\varepsilon = (\sigma_{ij}^\varepsilon)$ is the Cauchy stress tensor, $\mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon) = (e_{ij}^\varepsilon(\mathbf{u}^\varepsilon))$, with $e_{ij}^\varepsilon(\mathbf{u}^\varepsilon) := \frac{1}{2}(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon)$, is the linearized strain tensor, $\mathbf{D}^\varepsilon = (D_i^\varepsilon)$ is the electric displacement field, $\mathbf{B}^\varepsilon = (B_i^\varepsilon)$ is the magnetic induction field, $\mathcal{S}^\varepsilon$ is the thermodynamic entropy and $\mathbf{q}^\varepsilon = (q_i^\varepsilon)$ is the heat flow vector. $\mathbf{C} = (C_{ijk\ell})$, $\mathbf{P} = (P_{ijk})$, $\mathbf{R} = (R_{ijk})$, $\mathbf{X} = (X_{ij})$, $\mathbf{M} = (M_{ij})$, $\boldsymbol{\beta} = (\beta_{ij})$, $\boldsymbol{\alpha} = (\alpha_{ij})$, $\mathbf{p} = (p_i)$, $\mathbf{m} = (m_i)$, c_v and $\mathbf{K} = (K_{ij})$ represent, respectively, the elasticity tensor, the piezoelectric tensor, the piezomagnetic tensor, the dielectric permittivity tensor, the magnetic permeability tensor, the thermal stress tensor, the magneto-electric tensor, the pyroelectric vector, the pyromagnetic vector, the calorific capacity of the material and the thermal conductivity tensor. Moreover, we suppose the material properties of the (generally anisotropic) magneto-electro-thermo-elastic plate-like body under study to satisfy the usual symmetry, positivity and boundedness conditions, for which we refer to [9]. We recall here the most important hypotheses:

— The following symmetric matrix (see [9], [45])

$$\mathbb{M}^c := \begin{pmatrix} \mathbf{X} & \boldsymbol{\alpha} & \mathbf{p} \\ \boldsymbol{\alpha} & \mathbf{M} & \mathbf{m} \\ \mathbf{p}^T & \mathbf{m}^T & c_v \end{pmatrix}$$

is positive definite, which implies in particular that

$$X_{ij}a_j a_i + M_{ij}b_j b_i + 2\alpha_{ij}a_j b_i \geqslant C(a_i a_i + b_i b_i), \text{ for all } a_i, b_i \in \mathbb{R},\ C > 0. \tag{4.1}$$

This condition is verified when the components of the coupling constitutive parameters $\boldsymbol{\alpha}$, $\mathbf{p}$ and $\mathbf{m}$ are *small*, which is the case, for instance, for the usual $BaTiO_3$-$CoFe_2O_4$ composite (see Table 1 in [9]).

— Without loss of generality we assume that the mass tensor (see [49], [50])

$$\boldsymbol{\rho}^\varepsilon = \begin{pmatrix} \rho_1^\varepsilon & 0 & 0 \\ 0 & \rho_2^\varepsilon & 0 \\ 0 & 0 & \rho_3^\varepsilon \end{pmatrix}$$

is diagonal and positive definite [2]. Hence, $\rho_i^\varepsilon > 0$ and $\rho_i^\varepsilon \in L^\infty(\Omega^\varepsilon)$.

4.1.2 Governing Equations

The magneto-electro-thermo-elastic plate is subjected to body forces $\mathbf{f}^\varepsilon = (f_i^\varepsilon) : \Omega^\varepsilon \times (0,T) \to \mathbb{R}^3$, an electric charge density $\rho_e^\varepsilon : \Omega^\varepsilon \times (0,T) \to \mathbb{R}$ and heat source $r^\varepsilon : \Omega^\varepsilon \times (0,T) \to \mathbb{R}$. The state $\mathcal{U}^\varepsilon$ solves the following system of field equations:

$$\begin{cases} \boldsymbol{\rho}^\varepsilon \ddot{\mathbf{u}}^\varepsilon - \mathrm{div}^\varepsilon \boldsymbol{\sigma}^\varepsilon(\mathcal{U}^\varepsilon) = \mathbf{f}^\varepsilon & \text{in } \Omega^\varepsilon \times (0,T), \\ \mathrm{div}^\varepsilon \mathbf{D}^\varepsilon(\mathcal{U}^\varepsilon) = \rho_e^\varepsilon & \text{in } \Omega^\varepsilon \times (0,T), \\ \mathrm{div}^\varepsilon \mathbf{B}^\varepsilon(\mathcal{U}^\varepsilon) = 0 & \text{in } \Omega^\varepsilon \times (0,T), \\ \dot{\mathcal{S}}^\varepsilon(\mathcal{U}^\varepsilon) + \frac{1}{T_0}\mathrm{div}^\varepsilon \mathbf{q}^\varepsilon(\theta^\varepsilon) = r^\varepsilon & \text{in } \Omega^\varepsilon \times (0,T), \end{cases} \tag{4.2}$$

with $T_0 > 0$ a constant reference temperature. The boundary conditions are posed on $\partial\Omega^\varepsilon \times (0,T)$; we recall that $\partial\Omega^\varepsilon = \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \cup \Gamma_1^\varepsilon \cup \Gamma_0^\varepsilon$. For simplicity we consider homogeneous boundary conditions on $\Gamma_0^\varepsilon \times (0,T)$, concerning displacements and temperature, and non-homogeneous boundary conditions on $\widehat{\Gamma}^\varepsilon \times (0,T)$, concerning surface forces $\mathbf{g}^\varepsilon = (g_i^\varepsilon)$ and surface heat flow ϱ^ε. Hence, one has

$$\begin{cases} \boldsymbol{\sigma}^\varepsilon(\mathcal{U}^\varepsilon)\mathbf{n}^\varepsilon = \mathbf{g}^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon \times (0,T), & \mathbf{u}^\varepsilon = \mathbf{0} & \text{on } \Gamma_0^\varepsilon \times (0,T), \\ \mathbf{q}^\varepsilon(\theta^\varepsilon) \cdot \mathbf{n}^\varepsilon = \varrho^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon \times (0,T), & \theta^\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon \times (0,T). \end{cases} \tag{4.3}$$

2. Note that the mass tensor depends on ε.

As already shown in [63] (see, also, [53] for the case of a piezoelectric material with magnetic effects), we specify four possible sets of electromagnetic boundary conditions, leading to four different magneto-electro-thermo-elastic plate models:

$$(BC)_1 : \begin{cases} \mathbf{D}^\varepsilon(\mathcal{U}^\varepsilon)\cdot\mathbf{n}^\varepsilon = d^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon \times (0,T), \quad \varphi^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \times (0,T), \\ \mathbf{B}^\varepsilon(\mathcal{U}^\varepsilon)\cdot\mathbf{n}^\varepsilon = 0 & \text{on } \Gamma^\varepsilon \times (0,T), \quad \zeta^\varepsilon = \zeta^{\pm,\varepsilon} \quad \text{on } \Gamma_\pm^\varepsilon \times (0,T), \end{cases}$$

$$(BC)_2 : \begin{cases} \mathbf{D}^\varepsilon(\mathcal{U}^\varepsilon)\cdot\mathbf{n}^\varepsilon = 0 & \text{on } \Gamma^\varepsilon \times (0,T), \quad \varphi^\varepsilon = \varphi^{\pm,\varepsilon} \quad \text{on } \Gamma_\pm^\varepsilon \times (0,T), \\ \mathbf{B}^\varepsilon(\mathcal{U}^\varepsilon)\cdot\mathbf{n}^\varepsilon = b^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon \times (0,T), \quad \zeta^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \times (0,T), \end{cases}$$

$$(BC)_3 : \begin{cases} \mathbf{D}^\varepsilon(\mathcal{U}^\varepsilon)\cdot\mathbf{n}^\varepsilon = 0 & \text{on } \Gamma^\varepsilon \times (0,T), \quad \varphi^\varepsilon = \varphi^{\pm,\varepsilon} \quad \text{on } \Gamma_\pm^\varepsilon \times (0,T), \\ \mathbf{B}^\varepsilon(\mathcal{U}^\varepsilon)\cdot\mathbf{n}^\varepsilon = 0 & \text{on } \Gamma^\varepsilon \times (0,T), \quad \zeta^\varepsilon = \zeta^{\pm,\varepsilon} \quad \text{on } \Gamma_\pm^\varepsilon \times (0,T), \end{cases}$$

$$(BC)_4 : \begin{cases} \mathbf{D}^\varepsilon(\mathcal{U}^\varepsilon)\cdot\mathbf{n}^\varepsilon = d^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon \times (0,T), \quad \varphi^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \times (0,T), \\ \mathbf{B}^\varepsilon(\mathcal{U}^\varepsilon)\cdot\mathbf{n}^\varepsilon = b^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon \times (0,T), \quad \zeta^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \times (0,T), \end{cases}$$

where $\mathbf{n}^\varepsilon = (n_i^\varepsilon)$ is the outer unit normal vector to $\partial\Omega^\varepsilon$. Boundary conditions $(BC)_1$ lead to a plate which behaves simultaneously as a piezoelectric sensor and a piezomagnetic actuator, namely the *sensor-actuator* model. Boundary conditions $(BC)_2$ lead to a plate behaving simultaneously as a piezomagnetic sensor and a piezoelectric actuator, namely the *actuator-sensor* model. Boundary conditions $(BC)_3$ are associated with the *actuator* model, according to which the plate behaves as a piezoeletric and piezomagnetic actuator. Finally, boundary conditions $(BC)_4$ are related to the *sensor* model, whereby the plate behaves as a piezoelectric and piezomagnetic sensor.

The initial conditions are posed in Ω^ε. Let $\mathbf{u}_0^\varepsilon, \mathbf{u}_1^\varepsilon, \theta_0^\varepsilon$ be, respectively, the displacement, the velocity and the temperature at time $t = 0$; we have

$$\begin{cases} \mathbf{u}^\varepsilon(x^\varepsilon, 0) = \mathbf{u}^\varepsilon(0) = \mathbf{u}_0^\varepsilon & \text{in } \Omega^\varepsilon, \\ \dot{\mathbf{u}}^\varepsilon(x^\varepsilon, 0) = \dot{\mathbf{u}}^\varepsilon(0) = \mathbf{u}_1^\varepsilon & \text{in } \Omega^\varepsilon, \\ \theta^\varepsilon(x^\varepsilon, 0) = \theta^\varepsilon(0) = \theta_0^\varepsilon & \text{in } \Omega^\varepsilon. \end{cases}$$

Note that there is no need to impose initial conditions $\varphi_0^\varepsilon := \varphi^\varepsilon(0)$ and $\zeta_0^\varepsilon := \zeta^\varepsilon(0)$, since they are formally given by the solution of the following system of equations:

$$\begin{cases} \mathrm{div}^\varepsilon \mathbf{D}^\varepsilon(\mathcal{U}^\varepsilon)(0) = \partial_i^\varepsilon\Big(P_{ik\ell}e_{k\ell}^\varepsilon(\mathbf{u}_0^\varepsilon) - X_{ij}\partial_j^\varepsilon\varphi_0^\varepsilon - \alpha_{ij}\partial_j^\varepsilon\zeta_0^\varepsilon + p_i\theta_0^\varepsilon\Big) = \rho_e^\varepsilon, \\ \mathrm{div}^\varepsilon \mathbf{B}^\varepsilon(\mathcal{U}^\varepsilon)(0) = \partial_i^\varepsilon\Big(R_{ik\ell}e_{k\ell}^\varepsilon(\mathbf{u}_0^\varepsilon) - \alpha_{ij}\partial_j^\varepsilon\varphi_0^\varepsilon - M_{ij}\partial_j^\varepsilon\zeta_0^\varepsilon + m_i\theta_0^\varepsilon\Big) = 0, \end{cases} \tag{4.4}$$

equipped with suitable boundary conditions.

4.1.3 General Weak Formulation

In order to give a weak formulation of the problems introduced in the previous subsection, we follow Lions [43]. Given a certain magneto-electro-thermo-elastic state $\mathcal{U}^\varepsilon := (\mathbf{u}^\varepsilon, \varphi^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon)$, for all test functions [3] $\mathcal{V}^\varepsilon = (\mathbf{v}^\varepsilon, \psi^\varepsilon, \xi^\varepsilon, \eta^\varepsilon)$ and for any fixed

3. The space of test functions shall be precised case by case, according to the specific problem under study.

$t \in (0,T)$ we introduce the following bilinear form:

$$
\begin{aligned}
A^\varepsilon(\mathcal{U}^\varepsilon(t),\mathcal{V}^\varepsilon) :=\ & (\boldsymbol{\rho}^\varepsilon\ddot{\mathbf{u}}^\varepsilon,\mathbf{v}^\varepsilon) + c(\eta^\varepsilon,\dot{\mathbf{u}}^\varepsilon) + c_v(\dot{\theta}^\varepsilon,\eta^\varepsilon) - d(\eta^\varepsilon,\dot{\varphi}^\varepsilon) - e(\eta^\varepsilon,\dot{\zeta}^\varepsilon)+\\
& + a_u(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon) + b(\varphi^\varepsilon,\mathbf{v}^\varepsilon) - b(\psi^\varepsilon,\mathbf{u}^\varepsilon) + f(\zeta^\varepsilon,\mathbf{v}^\varepsilon) - f(\xi^\varepsilon,\mathbf{u}^\varepsilon)+\\
& - c(\theta^\varepsilon,\mathbf{v}^\varepsilon) + a_\varphi(\varphi^\varepsilon,\psi^\varepsilon) + a_\zeta(\zeta^\varepsilon,\xi^\varepsilon) + g(\zeta^\varepsilon,\psi^\varepsilon) + g(\varphi^\varepsilon,\xi^\varepsilon)+\\
& - d(\theta^\varepsilon,\psi^\varepsilon) - e(\theta^\varepsilon,\xi^\varepsilon) + a_\theta(\theta^\varepsilon,\eta^\varepsilon),
\end{aligned}
$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega^\varepsilon)$ and the bilinear forms $a_u(\cdot,\cdot)$, $a_\varphi(\cdot,\cdot)$, $a_\zeta(\cdot,\cdot)$, $a_\theta(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$, $d(\cdot,\cdot)$, $e(\cdot,\cdot)$, $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ are defined as follows:

$$
\begin{aligned}
a_u(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon) &:= \int_{\Omega^\varepsilon} C_{ijk\ell}e^\varepsilon_{k\ell}(\mathbf{u}^\varepsilon)e^\varepsilon_{ij}(\mathbf{v}^\varepsilon)dx^\varepsilon, & a_\varphi(\varphi^\varepsilon,\psi^\varepsilon) &:= \int_{\Omega^\varepsilon} X_{ij}\partial^\varepsilon_j\varphi^\varepsilon\partial^\varepsilon_i\psi^\varepsilon dx^\varepsilon,\\
a_\zeta(\zeta^\varepsilon,\xi^\varepsilon) &:= \int_{\Omega^\varepsilon} M_{ij}\partial^\varepsilon_j\zeta^\varepsilon\partial^\varepsilon_i\xi^\varepsilon dx^\varepsilon, & a_\theta(\theta^\varepsilon,\eta^\varepsilon) &:= \frac{1}{T_0}\int_{\Omega^\varepsilon} K_{ij}\partial^\varepsilon_j\theta^\varepsilon\partial^\varepsilon_i\eta^\varepsilon dx^\varepsilon,\\
b(\psi^\varepsilon,\mathbf{u}^\varepsilon) &:= \int_{\Omega^\varepsilon} P_{kij}\partial^\varepsilon_k\psi^\varepsilon e^\varepsilon_{ij}(\mathbf{u}^\varepsilon)dx^\varepsilon, & c(\eta^\varepsilon,\mathbf{u}^\varepsilon) &:= \int_{\Omega^\varepsilon} \eta^\varepsilon\beta_{ij}e^\varepsilon_{ij}(\mathbf{u}^\varepsilon)dx^\varepsilon,\\
d(\eta^\varepsilon,\varphi^\varepsilon) &:= \int_{\Omega^\varepsilon} \eta^\varepsilon p_k\partial^\varepsilon_k\varphi^\varepsilon dx^\varepsilon, & e(\eta^\varepsilon,\zeta^\varepsilon) &:= \int_{\Omega^\varepsilon} \eta^\varepsilon m_k\partial^\varepsilon_k\zeta^\varepsilon dx^\varepsilon,\\
f(\xi^\varepsilon,\mathbf{u}^\varepsilon) &:= \int_{\Omega^\varepsilon} R_{kij}\partial^\varepsilon_k\xi^\varepsilon e^\varepsilon_{ij}(\mathbf{u}^\varepsilon)dx^\varepsilon, & g(\zeta^\varepsilon,\psi^\varepsilon) &:= \int_{\Omega^\varepsilon} \alpha_{ij}\partial^\varepsilon_j\zeta^\varepsilon\partial^\varepsilon_i\psi^\varepsilon dx^\varepsilon.
\end{aligned}
$$

Besides, for a given magneto-electro-thermo-elastic state $\mathcal{U}^\varepsilon := (\mathbf{u}^\varepsilon,\varphi^\varepsilon,\zeta^\varepsilon,\theta^\varepsilon)$, we define the energy functional of the system:

$$
\begin{aligned}
\mathcal{E}^\varepsilon(t) :=\ & \frac{1}{2}\left\{(\boldsymbol{\rho}^\varepsilon\dot{\mathbf{u}}^\varepsilon,\dot{\mathbf{u}}^\varepsilon) + a_u(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) + a_\varphi(\varphi^\varepsilon,\varphi^\varepsilon) + a_\zeta(\zeta^\varepsilon,\zeta^\varepsilon) + (c^\varepsilon_v\theta^\varepsilon,\theta^\varepsilon)+\right.\\
& \left.-2d(\theta^\varepsilon,\varphi^\varepsilon) - 2e(\theta^\varepsilon,\zeta^\varepsilon) + 2g(\zeta^\varepsilon,\varphi^\varepsilon)\right\}.
\end{aligned}
$$

In the sequel we shall distinguish among the four variational evolution problems arising from the different possible boundary conditions presented in the previous subsection.

1) *The sensor-actuator model.* We let $\bar{\zeta}^\varepsilon := \zeta^\varepsilon - \widehat{\zeta}^\varepsilon$, where $\widehat{\zeta}^\varepsilon$ is a trace lifting in $H^1(\Omega^\varepsilon)$ of the magnetic boundary potentials $\zeta^{\pm,\varepsilon}$ acting on $\Gamma^\varepsilon_\pm$. The weak formulation of (4.2)-(4.3) with electromagnetic boundary conditions $(BC)_1$ takes the following form

$$
\left\{
\begin{array}{l}
\text{Find } \mathcal{U}^\varepsilon = (\mathbf{u}^\varepsilon,\varphi^\varepsilon,\bar{\zeta}^\varepsilon,\theta^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon,\Gamma^\varepsilon_0) \times H^1(\Omega^\varepsilon,\Gamma^\varepsilon_0)\times\\
\times H^1(\Omega^\varepsilon,\Gamma^\varepsilon_\pm) \times H^1(\Omega^\varepsilon,\Gamma^\varepsilon_0), \text{such that, for all}\\
\mathcal{V}^\varepsilon \in \mathbf{H}^1(\Omega^\varepsilon,\Gamma^\varepsilon_0) \times H^1(\Omega^\varepsilon,\Gamma^\varepsilon_0) \times H^1(\Omega^\varepsilon,\Gamma^\varepsilon_\pm) \times H^1(\Omega^\varepsilon,\Gamma^\varepsilon_0),\\
A^\varepsilon(\mathcal{U}^\varepsilon(t),\mathcal{V}^\varepsilon) = L^\varepsilon_1(\mathcal{V}^\varepsilon),\ t \in (0,T)
\end{array}
\right.
\tag{4.5}
$$

with initial conditions $(\mathbf{u}^\varepsilon_0,\mathbf{u}^\varepsilon_1,\theta^\varepsilon_0)$ and

$$
\begin{aligned}
L^\varepsilon_1(\mathcal{V}^\varepsilon) :=\ & (\mathbf{f}^\varepsilon,\mathbf{v}^\varepsilon) + (\mathbf{g}^\varepsilon,\mathbf{v}^\varepsilon)_{\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)} + (r^\varepsilon,\eta^\varepsilon) - \tfrac{1}{T_0}(\varrho^\varepsilon,\eta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} + (\rho^\varepsilon_e,\psi^\varepsilon)+\\
& -(d^\varepsilon,\psi^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} - a_\zeta(\widehat{\zeta}^\varepsilon,\xi^\varepsilon) - f(\widehat{\zeta}^\varepsilon,\mathbf{v}^\varepsilon) + e(\eta^\varepsilon,\partial_t\widehat{\zeta}^\varepsilon) - g(\widehat{\zeta}^\varepsilon,\psi^\varepsilon),
\end{aligned}
$$

As already shown in [9], for all weak solutions $\mathcal{U}^\varepsilon = (\mathbf{u}^\varepsilon, \varphi^\varepsilon, \bar{\zeta}^\varepsilon, \theta^\varepsilon)$ of problem (4.5), the energy solves the evolution equation

$$\dot{\mathcal{E}}^\varepsilon(t) + a_\theta(\theta^\varepsilon(t), \theta^\varepsilon(t)) = L^1_{\mathcal{E}}(t),$$

with $L^1_{\mathcal{E}}(t) := (\mathbf{f}^\varepsilon, \dot{\mathbf{u}}^\varepsilon) + (\mathbf{g}^\varepsilon, \dot{\mathbf{u}}^\varepsilon)_{\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)} + (\dot{\rho}^\varepsilon_e, \varphi^\varepsilon) - (\dot{d}^\varepsilon, \varphi^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} + (r^\varepsilon, \theta^\varepsilon) - \frac{1}{T_0}(\varrho^\varepsilon, \theta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} - a_\zeta(\partial_t \widehat{\zeta}^\varepsilon, \bar{\zeta}^\varepsilon) - f(\widehat{\zeta}^\varepsilon, \dot{\mathbf{u}}^\varepsilon) + e(\theta^\varepsilon, \partial_t \widehat{\zeta}^\varepsilon) - g(\partial_t \widehat{\zeta}^\varepsilon, \varphi^\varepsilon)$.

2) *The actuator-sensor model.* We let $\bar{\varphi}^\varepsilon := \varphi^\varepsilon - \widehat{\varphi}^\varepsilon$, where $\widehat{\varphi}^\varepsilon$ is a trace lifting in $H^1(\Omega^\varepsilon)$ of the electric boundary potentials $\varphi^{\pm,\varepsilon}$ acting on $\Gamma^\varepsilon_\pm$. The weak formulation of (4.2)-(4.3) with electromagnetic boundary conditions $(BC)_2$ takes the following form

$$\begin{cases} \text{Find } \mathcal{U}^\varepsilon = (\mathbf{u}^\varepsilon, \bar{\varphi}^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon, \Gamma^\varepsilon_0) \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_\pm) \times \\ \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_0) \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_0) \text{ such that, for all} \\ \mathcal{V}^\varepsilon \in \mathbf{H}^1(\Omega^\varepsilon, \Gamma^\varepsilon_0) \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_\pm) \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_0) \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_0), \\ A^\varepsilon(\mathcal{U}^\varepsilon(t), \mathcal{V}^\varepsilon) = L^\varepsilon_2(\mathcal{V}^\varepsilon), \ t \in (0, T) \end{cases} \tag{4.6}$$

with initial conditions $(\mathbf{u}^\varepsilon_0, \mathbf{u}^\varepsilon_1, \theta^\varepsilon_0)$ and

$$\begin{aligned} L^\varepsilon_2(\mathcal{V}^\varepsilon) := (\mathbf{f}^\varepsilon, \mathbf{v}^\varepsilon) + (\mathbf{g}^\varepsilon, \mathbf{v}^\varepsilon)_{\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)} + (r^\varepsilon, \eta^\varepsilon) - \tfrac{1}{T_0}(\varrho^\varepsilon, \eta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} + (\rho^\varepsilon_e, \psi^\varepsilon) + \\ -(b^\varepsilon, \xi^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} - a_\varphi(\widehat{\varphi}^\varepsilon, \psi^\varepsilon) - b(\widehat{\varphi}^\varepsilon, \mathbf{v}^\varepsilon) + d(\eta^\varepsilon, \partial_t \widehat{\varphi}^\varepsilon) - g(\xi^\varepsilon, \widehat{\varphi}^\varepsilon), \end{aligned}$$

For all weak solutions $\mathcal{U}^\varepsilon = (\mathbf{u}^\varepsilon, \bar{\varphi}^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon)$ of problem (4.6), the energy solves the evolution equation

$$\dot{\mathcal{E}}^\varepsilon(t) + a_\theta(\theta^\varepsilon(t), \theta^\varepsilon(t)) = L^2_{\mathcal{E}}(t),$$

with $L^2_{\mathcal{E}}(t) := (\mathbf{f}^\varepsilon, \dot{\mathbf{u}}^\varepsilon) + (\mathbf{g}^\varepsilon, \dot{\mathbf{u}}^\varepsilon)_{\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)} + (\dot{\rho}^\varepsilon_e, \varphi^\varepsilon) - (\dot{b}^\varepsilon, \zeta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} + (r^\varepsilon, \theta^\varepsilon) - \frac{1}{T_0}(\varrho^\varepsilon, \theta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} - a_\varphi(\partial_t \widehat{\varphi}^\varepsilon, \bar{\varphi}^\varepsilon) - b(\widehat{\varphi}^\varepsilon, \dot{\mathbf{u}}^\varepsilon) + d(\theta^\varepsilon, \partial_t \widehat{\varphi}^\varepsilon) - g(\zeta^\varepsilon, \partial_t \widehat{\varphi}^\varepsilon)$.

3) *The actuator model.* The weak formulation of (4.2)-(4.3) with electromagnetic boundary conditions $(BC)_3$ takes the following form

$$\begin{cases} \text{Find } \mathcal{U}^\varepsilon = (\mathbf{u}^\varepsilon, \bar{\varphi}^\varepsilon, \bar{\zeta}^\varepsilon, \theta^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon, \Gamma^\varepsilon_0) \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_\pm) \times \\ \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_\pm) \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_0) \text{ such that, for all} \\ \mathcal{V}^\varepsilon \in \mathbf{H}^1(\Omega^\varepsilon, \Gamma^\varepsilon_0) \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_\pm) \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_\pm) \times H^1(\Omega^\varepsilon, \Gamma^\varepsilon_0), \\ A^\varepsilon(\mathcal{U}^\varepsilon(t), \mathcal{V}^\varepsilon) = L^\varepsilon_3(\mathcal{V}^\varepsilon), \ t \in (0, T) \end{cases} \tag{4.7}$$

with initial conditions $(\mathbf{u}^\varepsilon_0, \mathbf{u}^\varepsilon_1, \theta^\varepsilon_0)$ and

$$\begin{aligned} L^\varepsilon_3(\mathcal{V}^\varepsilon) := (\mathbf{f}^\varepsilon, \mathbf{v}^\varepsilon) + (\mathbf{g}^\varepsilon, \mathbf{v}^\varepsilon)_{\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)} + (r^\varepsilon, \eta^\varepsilon) - \tfrac{1}{T_0}(\varrho^\varepsilon, \eta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} + (\rho^\varepsilon_e, \psi^\varepsilon) + \\ -a_\zeta(\widehat{\zeta}^\varepsilon, \psi^\varepsilon) - f(\widehat{\zeta}^\varepsilon, \mathbf{v}^\varepsilon) + e(\eta^\varepsilon, \partial_t \widehat{\zeta}^\varepsilon) - g(\widehat{\zeta}^\varepsilon, \psi^\varepsilon) + \\ -a_\varphi(\widehat{\varphi}^\varepsilon, \psi^\varepsilon) - b(\widehat{\varphi}^\varepsilon, \mathbf{v}^\varepsilon) + d(\eta^\varepsilon, \partial_t \widehat{\varphi}^\varepsilon) - g(\xi^\varepsilon, \widehat{\varphi}^\varepsilon). \end{aligned}$$

For all weak solutions $\mathcal{U}^\varepsilon = (\mathbf{u}^\varepsilon, \bar{\varphi}^\varepsilon, \bar{\zeta}^\varepsilon, \theta^\varepsilon)$ of problem (4.7), the energy solves the evolution equation

$$\dot{\mathcal{E}}^\varepsilon(t) + a_\theta(\theta^\varepsilon(t), \theta^\varepsilon(t)) = L^3_{\mathcal{E}}(t),$$

with $L^3_{\mathcal{E}}(t) := (\mathbf{f}^\varepsilon, \dot{\mathbf{u}}^\varepsilon) + (\mathbf{g}^\varepsilon, \dot{\mathbf{u}}^\varepsilon)_{\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)} + (\dot{\rho}^\varepsilon_e, \varphi^\varepsilon) + (r^\varepsilon, \theta^\varepsilon) - \frac{1}{T_0}(\varrho^\varepsilon, \theta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} - a_\zeta(\partial_t \widehat{\zeta}^\varepsilon, \bar{\zeta}^\varepsilon) - f(\widehat{\zeta}^\varepsilon, \dot{\mathbf{u}}^\varepsilon) + e(\theta^\varepsilon, \partial_t \widehat{\zeta}^\varepsilon) - g(\partial_t \widehat{\zeta}^\varepsilon, \varphi^\varepsilon) - a_\varphi(\partial_t \widehat{\varphi}^\varepsilon, \bar{\varphi}^\varepsilon) - b(\widehat{\varphi}^\varepsilon, \dot{\mathbf{u}}^\varepsilon) +$

$d(\theta^\varepsilon, \partial_t \widehat{\varphi}^\varepsilon) - g(\zeta^\varepsilon, \partial_t \widehat{\varphi}^\varepsilon)$.

4) *The sensor model.* The weak formulation of (4.2)-(4.3) with electromagnetic boundary conditions $(BC)_4$ takes the following form

$$\begin{cases} \text{Find } \mathcal{U}^\varepsilon = (\mathbf{u}^\varepsilon, \varphi^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon, \Gamma_0^\varepsilon) \times H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon) \times \\ \times H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon) \times H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon) \text{ such that, for all} \\ \mathcal{V}^\varepsilon \in \mathbf{H}^1(\Omega^\varepsilon, \Gamma_0^\varepsilon) \times H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon) \times H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon) \times H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon), \\ A^\varepsilon(\mathcal{U}^\varepsilon(t), \mathcal{V}^\varepsilon) = L_4^\varepsilon(\mathcal{V}^\varepsilon),\ t \in (0,T) \end{cases} \tag{4.8}$$

with initial conditions $(\mathbf{u}_0^\varepsilon, \mathbf{u}_1^\varepsilon, \theta_0^\varepsilon)$ and

$$L_4^\varepsilon(\mathcal{V}^\varepsilon) := (\mathbf{f}^\varepsilon, \mathbf{v}^\varepsilon) + (\mathbf{g}^\varepsilon, \mathbf{v}^\varepsilon)_{\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)} + (r^\varepsilon, \eta^\varepsilon) - \tfrac{1}{T_0}(\varrho^\varepsilon, \eta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} + (\rho_e^\varepsilon, \psi^\varepsilon) + \\ -(b^\varepsilon, \xi^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} - (d^\varepsilon, \psi^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)}$$

For all weak solutions $\mathcal{U}^\varepsilon = (\mathbf{u}^\varepsilon, \varphi^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon)$ of problem (4.8), the energy solves the evolution equation

$$\dot{\mathcal{E}}^\varepsilon(t) + a_\theta(\theta^\varepsilon(t), \theta^\varepsilon(t)) = L_{\mathcal{E}}^4(t),$$

with $L_{\mathcal{E}}^4(t) := (\mathbf{f}^\varepsilon, \dot{\mathbf{u}}^\varepsilon) + (\mathbf{g}^\varepsilon, \dot{\mathbf{u}}^\varepsilon)_{\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)} + (\dot{\rho}_e^\varepsilon, \varphi^\varepsilon) - (\dot{b}^\varepsilon, \zeta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} - (\dot{d}^\varepsilon, \varphi^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} + (r^\varepsilon, \theta^\varepsilon) - \frac{1}{T_0}(\varrho^\varepsilon, \theta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)}$.

So far, we have only assigned initial conditions on $\mathbf{u}^\varepsilon$ and θ^ε since, as pointed out in (4.4), initial conditions φ_0^ε and $\bar{\zeta}_0^\varepsilon$ are given by the solution of a suitable variational problem, according to the case under study; for instance, in the first case (sensor-actuator), we have

$$\begin{cases} \text{Find } (\varphi_0^\varepsilon, \bar{\zeta}_0^\varepsilon) \in H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon) \times H^1(\Omega^\varepsilon, \Gamma_\pm^\varepsilon) \text{ such that,} \\ \text{for all } (\psi^\varepsilon, \xi^\varepsilon) \in H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon) \times H^1(\Omega^\varepsilon, \Gamma_\pm^\varepsilon), \\ a_\varphi(\varphi_0^\varepsilon, \psi^\varepsilon) + a_\zeta(\bar{\zeta}_0^\varepsilon, \xi^\varepsilon) + g(\bar{\zeta}_0^\varepsilon, \psi^\varepsilon) + g(\varphi_0^\varepsilon, \xi^\varepsilon) = (\rho_e^\varepsilon(0), \psi^\varepsilon) + \\ -(d^\varepsilon(0), \psi^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)} - a_\zeta(\widehat{\zeta}_0^\varepsilon, \psi^\varepsilon) + f(\xi^\varepsilon, \mathbf{u}_0^\varepsilon) - e(\theta_0^\varepsilon, \xi^\varepsilon) - g(\widehat{\zeta}_0^\varepsilon, \psi^\varepsilon) + d(\theta_0^\varepsilon, \psi^\varepsilon); \end{cases} \tag{4.9}$$

in the other cases, analogous weak formulations can be given.

4.1.4 Existence, Uniqueness and Regularity

We state here a result of well-posedness for the problem in the sensor-actuator case, the other cases being analogous. The problem in its general setting has been dealt with in [9], to which we refer for the proof.

Theorem 4.1. *Suppose Ω^ε has Lipschitz-continuous boundary. Assume the following regularity properties on the initial data:*

$$(\mathbf{u}_0^\varepsilon, \mathbf{u}_1^\varepsilon, \theta_0^\varepsilon) \in \mathbf{H}^1(\Omega, \Gamma_0^\varepsilon) \times \mathbf{L}^2(\Omega^\varepsilon) \times H^1(\Omega, \Gamma_0^\varepsilon),$$

the following regularity properties on source and boundary values:

$$\begin{cases} \mathbf{f}^\varepsilon \in L^2(0,T;\mathbf{L}^2(\Omega^\varepsilon)), \\ \rho_e^\varepsilon \in H^1(0,T;L^2(\Omega^\varepsilon)) \cap C^0([0,T];L^2(\Omega^\varepsilon)), \\ r^\varepsilon \in L^2(0,T;L^2(\Omega^\varepsilon)), \\ \mathbf{g}^\varepsilon \in H^2(0,T;\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)) \cap C^1([0,T];\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)), \\ d^\varepsilon \in H^1(0,T;L^2(\widehat{\Gamma}^\varepsilon)) \cap C^0([0,T];L^2(\widehat{\Gamma}^\varepsilon)), \\ b^\varepsilon \in H^1(0,T;L^2(\Gamma^\varepsilon)) \cap C^0([0,T];L^2(\Gamma^\varepsilon)), \\ \varrho^\varepsilon \in L^2(0,T;L^2(\widehat{\Gamma}^\varepsilon)), \end{cases}$$

and the following compatibility conditions:

$$\begin{cases} \mathbf{g}^\varepsilon(0) = \boldsymbol{\sigma}(\mathbf{u}_0^\varepsilon,\varphi_0^\varepsilon,\zeta_0^\varepsilon,\theta_0^\varepsilon)\mathbf{n}^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon, \\ d^\varepsilon(0) = \mathbf{D}^\varepsilon(\mathbf{u}_0^\varepsilon,\varphi_0^\varepsilon,\zeta_0^\varepsilon,\theta_0^\varepsilon)\cdot\mathbf{n}^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon, \\ b^\varepsilon(0) = \mathbf{B}^\varepsilon(\mathbf{u}_0^\varepsilon,\varphi_0^\varepsilon,\zeta_0^\varepsilon,\theta_0^\varepsilon)\cdot\mathbf{n}^\varepsilon & \text{on } \Gamma^\varepsilon, \\ \varrho^\varepsilon(0) = \mathbf{q}^\varepsilon(\theta_0^\varepsilon)\cdot\mathbf{n}^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon. \end{cases} \tag{4.10}$$

Then, problem (4.5) admits a unique solution $(\mathbf{u}^\varepsilon,\varphi^\varepsilon,\bar{\zeta}^\varepsilon,\theta^\varepsilon)$ *such that*

$$\begin{cases} \mathbf{u}^\varepsilon \in L^2(0,T;\mathbf{H}^1(\Omega^\varepsilon,\Gamma_0^\varepsilon)) \cap C^0([0,T];\mathbf{L}^2(\Omega^\varepsilon)), \\ \dot{\mathbf{u}}^\varepsilon \in L^2(0,T;\mathbf{L}^2(\Omega^\varepsilon)), \\ \boldsymbol{\rho}^\varepsilon\ddot{\mathbf{u}}^\varepsilon \in L^2(0,T;\mathbf{H}^*(\Omega^\varepsilon,\Gamma_0^\varepsilon)), \\ \varphi^\varepsilon \in L^2(0,T;H^1(\Omega^\varepsilon,\Gamma_0^\varepsilon)), \\ \bar{\zeta}^\varepsilon \in L^2(0,T;H^1(\Omega^\varepsilon,\Gamma_\pm^\varepsilon)), \\ \theta^\varepsilon \in L^2(0,T;H^1(\Omega^\varepsilon,\Gamma_0^\varepsilon)), \\ c_v\dot{\theta}^\varepsilon + \boldsymbol{\beta}:\mathbf{e}^\varepsilon(\dot{\mathbf{u}}^\varepsilon) - \mathbf{p}\cdot\nabla^\varepsilon\dot{\varphi}^\varepsilon - \mathbf{m}\cdot\nabla^\varepsilon\dot{\bar{\zeta}}^\varepsilon \in L^2(0,T;H^*(\Omega^\varepsilon,\Gamma_0^\varepsilon)). \end{cases}$$

One can also prove the following result.

Theorem 4.2. *Besides (4.10), suppose the following further conditions are satisfied by the initial data and the domain:*

(i)
$$\begin{cases} \mathbf{u}_0^\varepsilon \in \mathbf{H}^2(\Omega^\varepsilon) \cap \mathbf{H}^1(\Omega^\varepsilon,\Gamma_0^\varepsilon), \\ \mathbf{u}_1^\varepsilon \in \mathbf{H}^1(\Omega^\varepsilon,\Gamma_0^\varepsilon), \\ \theta_0^\varepsilon \in H^2(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon,\Gamma_0^\varepsilon), \end{cases}$$

(ii) $\mathbf{u}_0^\varepsilon$, θ_0^ε *and the domain* Ω^ε *are such that problem (4.9) admits a solution* $(\varphi_0^\varepsilon,\bar{\zeta}_0^\varepsilon) \in H^2(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon,\Gamma_0^\varepsilon) \times H^2(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon,\Gamma_\pm^\varepsilon)$.

Secondly, assume the following further regularity properties on the source and boundary values:

$$\begin{cases} \mathbf{f}^\varepsilon \in H^1(0,T;\mathbf{L}^2(\Omega^\varepsilon)) \cap C^0([0,T];\mathbf{L}^2(\Omega^\varepsilon)), \\ \rho_e \in H^2(0,T;L^2(\Omega^\varepsilon)) \cap C^1([0,T];L^2(\Omega^\varepsilon)), \\ r \in H^1(0,T;L^2(\Omega^\varepsilon)) \cap C^0([0,T];L^2(\Omega^\varepsilon)), \\ \mathbf{g}^\varepsilon \in H^2(0,T;\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)) \cap C^1([0,T];\mathbf{L}^2(\widehat{\Gamma}^\varepsilon)), \\ d^\varepsilon \in H^2(0,T;L^2(\widehat{\Gamma}^\varepsilon)) \cap C^1([0,T];L^2(\widehat{\Gamma}^\varepsilon)), \\ b^\varepsilon \in H^2(0,T;L^2(\Gamma^\varepsilon)) \cap C^1([0,T];L^2(\Gamma^\varepsilon)), \\ \varrho^\varepsilon \in H^1(0,T;L^2(\widehat{\Gamma}^\varepsilon)) \cap C^0([0,T];L^2(\widehat{\Gamma}^\varepsilon)), \end{cases}$$

and on the constitutive parameters:

$$\begin{cases} C_{ijk\ell} \in W^{1,\infty}(\Omega^\varepsilon), \\ P_{ijk} \in W^{1,\infty}(\Omega^\varepsilon), \\ R_{ijk} \in W^{1,\infty}(\Omega^\varepsilon), \\ \beta_{ij} \in W^{1,\infty}(\Omega^\varepsilon), \\ K_{ij} \in W^{1,\infty}(\Omega^\varepsilon). \end{cases}$$

Then, problem (4.5) admits a unique solution $(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \bar{\zeta}^\varepsilon, \theta^\varepsilon)$ *such that*

$$\begin{cases} \mathbf{u}^\varepsilon \in C^0([0,T]; \mathbf{H}^1(\Omega^\varepsilon, \Gamma_0^\varepsilon)) \cap C^1([0,T]; \mathbf{L}^2(\Omega^\varepsilon)), \\ \dot{\mathbf{u}}^\varepsilon \in L^2(0,T; \mathbf{H}^1(\Omega^\varepsilon, \Gamma_0^\varepsilon)), \\ \ddot{\mathbf{u}}^\varepsilon \in L^2(0,T; \mathbf{L}^2(\Omega^\varepsilon)), \\ \varphi^\varepsilon \in H^1(0,T; H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon)) \cap C^0([0,T]; H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon)), \\ \bar{\zeta}^\varepsilon \in H^1(0,T; H^1(\Omega^\varepsilon, \Gamma_\pm^\varepsilon)) \cap C^0([0,T]; H^1(\Omega^\varepsilon, \Gamma_\pm^\varepsilon)), \\ \theta^\varepsilon \in H^1(0,T; H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon)) \cap C^0([0,T]; H^1(\Omega^\varepsilon, \Gamma_0^\varepsilon)). \end{cases}$$

Let us explicitly remark that condition (ii) is automatically satisfied when Ω^ε is convex (see, e.g., [32]).

We just provide the main ideas of the proof, which is always based on the Faedo-Galerkin method, as in the case of Theorem 4.1 (see Thm. 2.1 in [9]). For typographical reasons, from here until the end of the current subsection we reckon to fix a value of ε and drop this index from all the mathematical objects involved. As compared with the procedure carried out in [9], according to the further regularity hypotheses on the initial data, here we introduce bases $\{\mathbf{v}_k\}_{k=1}^\infty$, $\{\eta_k\}_{k=1}^\infty$ and $\{\xi_k\}_{k=1}^\infty$ for spaces $\mathbf{H}^2(\Omega) \cap \mathbf{H}^1(\Omega, \Gamma_0)$, $H^2(\Omega) \cap H^1(\Omega, \Gamma_0)$ and $H^2(\Omega) \cap H^1(\Omega, \Gamma_\pm)$, respectively. At this stage, the proof can be split in two parts. In the first part, we fix $p \in \mathbb{N}$ and define $\mathbf{u}_p(t)$, $\varphi_p(t)$, $\zeta_p(t)$ and $\theta_p(t)$ as linear combinations with time-dependent coefficients of $\mathbf{v}_1, \ldots, \mathbf{v}_p$, $\eta_1, \ldots, \eta_p$, $\xi_1, \ldots, \xi_p$ and again $\eta_1, \ldots, \eta_p$, respectively. Then, we require such coefficients to satisfy the following strong convergences:

$$\begin{cases} \mathbf{u}_p(0) \to \mathbf{u}_0 & \text{in } \mathbf{H}^2(\Omega) \cap \mathbf{H}^1(\Omega, \Gamma_0), \\ \dot{\mathbf{u}}_p(0) \to \mathbf{u}_1 & \text{in } \mathbf{H}^1(\Omega, \Gamma_0), \\ \varphi_p(0) \to \varphi_0 & \text{in } H^2(\Omega) \cap H^1(\Omega, \Gamma_0), \\ \zeta_p(0) \to \zeta_0 & \text{in } H^2(\Omega) \cap H^1(\Omega, \Gamma_\pm), \\ \theta_p(0) \to \theta_0 & \text{in } H^2(\Omega) \cap H^1(\Omega, \Gamma_0), \end{cases} \tag{4.11}$$

where (φ_0, ζ_0) is the solution of (4.9). Thus, as in [45], the mapping $t \mapsto (\mathbf{u}_p(t), \varphi_p(t), \zeta_p(t), \theta_p(t))$ is determined as the solution of a well-posed finite-dimensional system of ordinary differential and algebraic equations, with initial conditions satisfying (4.11). Finally, weak convergence results as $p \to +\infty$ are established using procedures and estimates completely analogous to those carried out in steps (i), (ii) and (iii) of the proof of Theorem 4.3: that is to say, respectively, finding a bound on the external power depending on the energy, then finding a bound on the initial energy and eventually applying Gronwall's lemma to find a bound on the energy at time t. The weak limits provide the sought solution of (4.5) in certain spaces of the form $L^2(0,T;H)$ with H a suitable Hilbert space. In the second part of the proof, in-time regularity of the solution is increased, by applying exactly the same arguments of the first part to the list of unknowns $\partial_t \mathcal{U} := (\partial_t \mathbf{u}, \partial_t \varphi, \partial_t \bar{\zeta}, \partial_t \theta)$. One finally finds analogous bounds on $\mathcal{E}(\partial_t \mathcal{U}_p(t))$ and on the quadratic form $a_\theta(\partial_t \theta_p(t), \partial_t \theta_p(t))$, where $\mathcal{E}(\partial_t \mathcal{U}_p(t))$ denotes the energy associated with $\partial_t \mathcal{U}_p(t)$. Hence, further weak convergence results as $p \to +\infty$ are obtained and, as a consequence, one finds a solution with further in-time regularity properties.

It is worth to point out that the regularity hypotheses on initial data and constitutive parameters are essential to obtain bounds on $\mathcal{E}(\partial_t \mathcal{U}_p(0))$ (in particular, on $|\partial_{tt} \mathbf{u}_p(0)|_{0,\Omega}$ and $|\partial_t \theta_p(0)|_{0,\Omega}$), so that one can apply Gronwall's lemma to get an estimate on $\mathcal{E}(\partial_t \mathcal{U}_p(t))$. Indeed, when looking for a bound of $|\partial_{tt} \mathbf{u}_p(0)|_{0,\Omega}$, one has to deal with a term of the form

$$\left|\operatorname{div}\left(\mathbf{Ce}(\mathbf{u}_p(0)) + \mathbf{P}^T \nabla \varphi_p(0) + \mathbf{R}^T \nabla \bar{\zeta}_p(0) - \boldsymbol{\beta} \theta_p(0)\right)\right|_{0,\Omega},$$

and so the additional assumptions ensure this $L^2(\Omega)$-norm to be well-defined. Similarly, a term of the form

$$\left|\operatorname{div} \mathbf{K} \nabla \theta_p(0)\right|_{0,\Omega}$$

has to be dealt with when looking for a bound on $|\partial_t \theta_p(0)|_{0,\Omega}$.

4.1.5 Scaled Evolution Problems

In order to perform an asymptotic analysis, we need to transform problems (4.5), (4.6), (4.7), (4.8), posed on a variable domain Ω^ε, onto problems posed on a fixed domain Ω (independent of ε). We suppose that the thickness of the plate h^ε depends linearly on ε, so that $h^\varepsilon = \varepsilon h$. Accordingly, we let

$$\begin{aligned}
&\Omega := \omega \times (-h, h), \\
&\Gamma_0 := \gamma_0 \times (-h, h), \ \ \Gamma_1 := \gamma_1 \times (-h, h), \\
&\Gamma_\pm := \omega \times \{\pm h\}, \ \ \widehat{\Gamma} := \Gamma_\pm \cup \Gamma_1.
\end{aligned}$$

Moreover, for $\Xi \subset \partial\Omega$, we define the following functional spaces

$$\begin{aligned}
H^1(\Omega, \Xi) &:= \left\{v \in H^1(\Omega);\ v = 0 \text{ on } \Xi\right\}, \\
\mathbf{H}^1(\Omega, \Xi) &:= \left\{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega);\ \mathbf{v} = \mathbf{0} \text{ on } \Xi\right\}.
\end{aligned}$$

Hence, we apply the following change of variables (see [14]):

$$\pi^\varepsilon : x \equiv (\tilde{x}, x_3) \in \overline{\Omega} \mapsto x^\varepsilon \equiv (\tilde{x}, \varepsilon x_3) \in \overline{\Omega}^\varepsilon, \text{ with } \tilde{x} = (x_\alpha).$$

By using the bijection π^ε, one has $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$.

Let us suppose that the data verify the following scaling assumptions:

$$\begin{array}{llll}
f_\alpha^\varepsilon(x^\varepsilon, t) = f_\alpha(x, t), & f_3^\varepsilon(x^\varepsilon, t) = \varepsilon f_3(x, t), & x \in \Omega, & \\
g_\alpha^\varepsilon(x^\varepsilon, t) = g_\alpha(x, t), & g_3^\varepsilon(x^\varepsilon, t) = \varepsilon g_3(x, t), & x \in \Gamma_1, & \\
g_\alpha^\varepsilon(x^\varepsilon, t) = \varepsilon g_\alpha(x, t), & g_3^\varepsilon(x^\varepsilon, t) = \varepsilon^2 g_3(x, t), & x \in \Gamma_\pm, & \\
\rho_e^\varepsilon(x^\varepsilon, t) = \rho_e(x, t), & r^\varepsilon(x^\varepsilon, t) = r(x, t), & x \in \Omega & \\
d^\varepsilon(x^\varepsilon, t) = d(x, t), & b^\varepsilon(x^\varepsilon, t) = b(x, t), & \varrho^\varepsilon(x^\varepsilon, t) = \varrho(x, t), & x \in \Gamma_1, \\
d^\varepsilon(x^\varepsilon, t) = \varepsilon d(x, t), & b^\varepsilon(x^\varepsilon, t) = \varepsilon b(x, t), & \varrho^\varepsilon(x^\varepsilon, t) = \varepsilon \varrho(x, t), & x \in \Gamma_\pm.
\end{array}$$

We assume the following scalings for the mass densities ρ_i^ε, as in [45] (see also [7]):

$$\rho_\alpha^\varepsilon(x^\varepsilon) = \rho(x), \quad \rho_3^\varepsilon(x^\varepsilon) = \varepsilon^2 \rho(x), \quad x \in \Omega.$$

Remark 4.1. The in-plane and transversal components of the mass tensor are scaled differently. These assumptions aim at obtaining a scaled evolution problem that couples the three components of the displacement field. In particular [7], the ε^2 dependence of ρ_3^ε allows, as an example, for an upward shift in the purely elastic transversal vibration frequencies of the plate as the scaling parameter goes to zero. Thus, the limit model is sensitive to inertia effects along the transversal direction, as it will be shown in the presentation of the flexural problems.

We distinguish the four cases of study for what concerns the scalings of the unknowns and test functions. In particular, since the mechanical and thermal loads and boundary conditions remain unvaried in any case, the scalings of the unknown displacements u_i^ε and temperature θ^ε and their associated test functions shall always be

$$\begin{aligned}
u_\alpha^\varepsilon(x^\varepsilon,t) &= u_\alpha(\varepsilon)(x,t) && \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T),\\
u_3^\varepsilon(x^\varepsilon,t) &= \varepsilon^{-1}u_3(\varepsilon)(x,t) && \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T),\\
\theta^\varepsilon(x^\varepsilon,t) &= \theta(\varepsilon)(x,t) && \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T),
\end{aligned}$$

hence the associated scaled strain tensor field $\boldsymbol{\kappa}(\varepsilon) = (\kappa_{ij}(\varepsilon))$, with $\kappa_{ij}(\varepsilon) \in L^2(\Omega)$ and scaled temperature gradient $\boldsymbol{\gamma}(\varepsilon) = (\gamma_i(\varepsilon))$, with $\gamma_i(\varepsilon) \in L^2(\Omega)$ are always given by

$$\begin{aligned}
&\kappa_{\alpha\beta}(\varepsilon) := e_{\alpha\beta}(\mathbf{u}(\varepsilon)), \quad \kappa_{\alpha 3}(\varepsilon) := \tfrac{1}{\varepsilon}e_{\alpha 3}(\mathbf{u}(\varepsilon)), \quad \kappa_{33}(\varepsilon) := \tfrac{1}{\varepsilon^2}e_{33}(\mathbf{u}(\varepsilon)),\\
&\gamma_\alpha(\varepsilon) := \partial_\alpha\theta(\varepsilon), \qquad \gamma_3(\varepsilon) := \tfrac{1}{\varepsilon}\partial_3\theta(\varepsilon).
\end{aligned}$$

Due to the different electromagnetic source terms and boundary conditions (see, e.g., [63]), the scalings related to the electric and magnetic potentials φ^ε and ζ^ε shall vary throughout the asymptotic procedure; of course, the same holds for the scalings of the corresponding test functions. The scaled gradients of the electric and magnetic potentials will be denoted, respectively, by $\boldsymbol{\tau}(\varepsilon) = (\tau_i(\varepsilon))$, with $\tau_i(\varepsilon) \in L^2(\Omega)$, and $\boldsymbol{\chi}(\varepsilon) = (\chi_i(\varepsilon))$, with $\chi_i(\varepsilon) \in L^2(\Omega)$.

In general, with an arbitrary magneto-electro-thermo-elastic state $\mathcal{V} = (\mathbf{v}, \psi, \xi, \eta)$, we associate, respectively, the tensor field $\boldsymbol{\kappa}(\varepsilon; \mathbf{v}) = (\kappa_{ij}(\varepsilon; \mathbf{v}))$ and vector fields $\boldsymbol{\tau}(\varepsilon;\psi) = (\tau_i(\varepsilon;\psi))$, $\boldsymbol{\chi}(\varepsilon;\xi) = (\chi_i(\varepsilon;\xi))$ and $\boldsymbol{\gamma}(\varepsilon;\eta) = (\gamma_i(\varepsilon;\eta))$.

1) *The sensor-actuator model.* With the unknown magneto-electro-thermo-elastic state $\mathcal{U}^\varepsilon \in \mathbf{H}^1(\Omega^\varepsilon,\Gamma_0^\varepsilon) \times H^1(\Omega^\varepsilon,\Gamma_0^\varepsilon) \times H^1(\Omega^\varepsilon,\Gamma_\pm^\varepsilon) \times H^1(\Omega^\varepsilon,\Gamma_0^\varepsilon)$, we associate the scaled magneto-electro-thermo-elastic state $\mathcal{U}(\varepsilon) := (\mathbf{u}(\varepsilon), \varphi(\varepsilon), \bar\zeta(\varepsilon), \theta(\varepsilon)) \in \mathbf{H}^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_\pm) \times H^1(\Omega,\Gamma_0)$, where

$$\begin{aligned}
\varphi^\varepsilon(x^\varepsilon,t) &= \varphi(\varepsilon)(x,t) && \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T),\\
\bar\zeta^\varepsilon(x^\varepsilon,t) &= \varepsilon\bar\zeta(\varepsilon)(x,t) && \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T).
\end{aligned} \tag{4.12}$$

Moreover, with these scalings we associate, respectively,

$$\begin{aligned}
\tau_\alpha(\varepsilon) &:= \partial_\alpha\varphi(\varepsilon), & \tau_3(\varepsilon) &:= \tfrac{1}{\varepsilon}\partial_3\varphi(\varepsilon),\\
\chi_\alpha(\varepsilon) &:= \varepsilon\partial_\alpha\zeta(\varepsilon), & \chi_3(\varepsilon) &:= \partial_3\zeta(\varepsilon),
\end{aligned}$$

We let $\bar{\boldsymbol{\chi}}(\varepsilon) := \boldsymbol{\chi}(\varepsilon; \bar\zeta(\varepsilon))$ and $\widehat{\boldsymbol{\chi}} := \nabla\widehat{\zeta}$.

We can now reformulate the problem on the fixed domain Ω. It follows that for every $\varepsilon > 0$ the scaled magneto-electro-thermo-elastic state $\mathcal{U}(\varepsilon)$ is the unique solution to the scaled problem:

$$\left\{\begin{array}{l}\text{Find } \mathcal{U}(\varepsilon) \in \mathbf{H}^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_\pm) \times H^1(\Omega,\Gamma_0),\\ \text{such that, for all } \mathcal{V} \in \mathbf{H}^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_\pm) \times H^1(\Omega,\Gamma_0),\\ A(\varepsilon)(\mathcal{U}(\varepsilon)(t),\mathcal{V}) = L_1(\varepsilon)(\mathcal{V}),\ t \in (0,T),\end{array}\right. \tag{4.13}$$

with initial conditions $(\mathbf{u}_0, \mathbf{u}_1, \theta_0)$, where

$$\begin{aligned}A(\varepsilon)(\mathcal{U}(\varepsilon)(t),\mathcal{V}) &:= (\rho\ddot{\mathbf{u}}(\varepsilon),\mathbf{v}) + c(\varepsilon)(\eta,\dot{\mathbf{u}}(\varepsilon)) + c_v(\dot{\theta}(\varepsilon),\eta) - d(\varepsilon)(\eta,\dot{\varphi}(\varepsilon)) +\\ &\quad -e(\varepsilon)(\eta,\dot{\zeta}(\varepsilon)) + a_u(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{v}) + b(\varepsilon)(\varphi(\varepsilon),\mathbf{v}) - b(\varepsilon)(\psi,\mathbf{u}(\varepsilon)) +\\ &\quad +f(\varepsilon)(\zeta(\varepsilon),\mathbf{v}) - f(\varepsilon)(\xi,\mathbf{u}(\varepsilon)) - c(\varepsilon)(\theta(\varepsilon),\mathbf{v}) + a_\varphi(\varepsilon)(\varphi(\varepsilon),\psi) +\\ &\quad +a_\zeta(\varepsilon)(\zeta(\varepsilon),\xi) + g(\varepsilon)(\zeta(\varepsilon),\psi) + g(\varepsilon)(\varphi(\varepsilon),\xi) - d(\varepsilon)(\theta(\varepsilon),\psi) +\\ &\quad -e(\varepsilon)(\theta(\varepsilon),\xi) + a_\theta(\varepsilon)(\theta(\varepsilon),\eta),\\ L_1(\varepsilon)(\mathcal{V}) &:= (\mathbf{f},\mathbf{v}) + (\mathbf{g},\mathbf{v})_{\mathbf{L}^2(\hat{\Gamma})} + (r,\eta) - \tfrac{1}{T_0}(\varrho,\eta)_{L^2(\hat{\Gamma})} + (\rho_e,\psi) - (d,\psi)_{L^2(\hat{\Gamma})} +\\ &\quad -a_\zeta(\varepsilon)(\hat{\zeta},\xi) - f(\varepsilon)(\hat{\zeta},\mathbf{v}) + e(\varepsilon)(\eta,\partial_t\hat{\zeta}) - g(\varepsilon)(\hat{\zeta},\psi).\end{aligned}$$

The new bilinear forms are defined as follows:

$$\begin{aligned}
a_u(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{v}) &:= \int_\Omega C_{ijk\ell}\kappa_{k\ell}(\varepsilon)\kappa_{ij}(\varepsilon;\mathbf{v})dx, & a_\varphi(\varepsilon)(\varphi(\varepsilon),\psi) &:= \int_\Omega X_{ij}\tau_j(\varepsilon)\tau_i(\varepsilon;\psi)dx,\\
a_\zeta(\varepsilon)(\zeta(\varepsilon),\xi) &:= \int_\Omega M_{ij}\chi_j(\varepsilon)\chi_i(\varepsilon;\xi)dx, & a_\theta(\varepsilon)(\theta(\varepsilon),\eta) &:= \frac{1}{T_0}\int_\Omega K_{ij}\gamma_j(\varepsilon)\gamma_i(\varepsilon;\eta)dx,\\
b(\varepsilon)(\psi,\mathbf{u}(\varepsilon)) &:= \int_\Omega P_{kij}\tau_k(\varepsilon;\psi)\kappa_{ij}(\varepsilon)dx, & c(\varepsilon)(\eta,\mathbf{u}(\varepsilon)) &:= \int_\Omega \eta\beta_{ij}\kappa_{ij}(\varepsilon)dx,\\
d(\varepsilon)(\eta,\varphi(\varepsilon)) &:= \int_\Omega \eta p_k\tau_k(\varepsilon)dx, & e(\varepsilon)(\eta,\zeta(\varepsilon)) &:= \int_\Omega \eta m_k\chi_k(\varepsilon)dx,\\
f(\varepsilon)(\xi,\mathbf{u}(\varepsilon)) &:= \int_\Omega R_{kij}\chi_k(\varepsilon;\xi)\kappa_{ij}(\varepsilon)dx, & g(\varepsilon)(\zeta(\varepsilon),\psi) &:= \int_\Omega \alpha_{ij}\chi_j(\varepsilon)\tau_i(\varepsilon;\psi)dx.
\end{aligned}$$

We denote by $\mathcal{E}(\varepsilon)(t)$ the scaled energy of the system associated with a weak solution $\mathcal{U}(\varepsilon) = (\mathbf{u}(\varepsilon),\varphi(\varepsilon),\zeta(\varepsilon),\theta(\varepsilon))$:

$$\begin{aligned}\mathcal{E}(\varepsilon)(t) := \frac{1}{2}\big\{&(\rho\dot{\mathbf{u}}(\varepsilon),\dot{\mathbf{u}}(\varepsilon)) + a_u(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon)) + a_\varphi(\varepsilon)(\varphi(\varepsilon)+\\ &+\varphi(\varepsilon)) + a_\zeta(\varepsilon)(\zeta(\varepsilon),\zeta(\varepsilon)) + (c_v\theta(\varepsilon),\theta(\varepsilon)) +\\ &-2d(\varepsilon)(\theta(\varepsilon),\varphi(\varepsilon)) - 2e(\varepsilon)(\theta(\varepsilon),\zeta(\varepsilon)) + 2g(\varepsilon)(\zeta(\varepsilon),\varphi(\varepsilon))\big\}.\end{aligned}$$

Following [9], it is easy to prove that, for all weak solutions $\mathcal{U}(\varepsilon) := (\mathbf{u}(\varepsilon),\varphi(\varepsilon),\bar{\zeta}(\varepsilon),\theta(\varepsilon))$ of problem (4.13), the scaled energy solves the evolution equation

$$\dot{\mathcal{E}}(\varepsilon)(t) + a_\theta(\varepsilon)(\theta(\varepsilon),\theta(\varepsilon)) = L^1_{\mathcal{E}}(\varepsilon)(t), \tag{4.14}$$

with

$$\begin{aligned}L^1_{\mathcal{E}}(\varepsilon)(t) := &(\mathbf{f},\dot{\mathbf{u}}(\varepsilon)) + (\mathbf{g},\dot{\mathbf{u}}(\varepsilon))_{\mathbf{L}^2(\hat{\Gamma})} + (\dot{\rho}_e,\varphi(\varepsilon)) - (\dot{d},\varphi(\varepsilon))_{L^2(\hat{\Gamma})} +\\ &+(r,\theta(\varepsilon)) - \tfrac{1}{T_0}(\varrho,\theta(\varepsilon))_{L^2(\hat{\Gamma})} - a_\zeta(\varepsilon)(\partial_t\hat{\zeta},\bar{\zeta}(\varepsilon)) - f(\varepsilon)(\hat{\zeta},\dot{\mathbf{u}}(\varepsilon)) +\\ &+e(\varepsilon)(\theta(\varepsilon),\partial_t\hat{\zeta}) - g(\varepsilon)(\partial_t\hat{\zeta},\varphi(\varepsilon)).\end{aligned} \tag{4.15}$$

2) *The actuator-sensor model.* We assume the following scalings for φ^ε and ζ^ε:

$$\begin{aligned} \bar{\varphi}^\varepsilon(x^\varepsilon,t) &= \varepsilon\bar{\varphi}(\varepsilon)(x,t) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T), \\ \zeta^\varepsilon(x^\varepsilon,t) &= \zeta(\varepsilon)(x,t) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T). \end{aligned} \tag{4.16}$$

Moreover, the components of the scaled vector fields $\boldsymbol{\tau}(\varepsilon) = (\tau_i(\varepsilon))$ and $\boldsymbol{\chi}(\varepsilon) = (\chi_i(\varepsilon))$ are now defined by

$$\begin{aligned} \tau_\alpha(\varepsilon) &\coloneqq \varepsilon\partial_\alpha\varphi(\varepsilon), \quad \tau_3(\varepsilon) \coloneqq \partial_3\varphi(\varepsilon), \\ \chi_\alpha(\varepsilon) &\coloneqq \partial_\alpha\zeta(\varepsilon), \quad \chi_3(\varepsilon) \coloneqq \tfrac{1}{\varepsilon}\partial_3\zeta(\varepsilon). \end{aligned}$$

We let $\bar{\boldsymbol{\tau}}(\varepsilon) \coloneqq \boldsymbol{\tau}(\varepsilon;\bar{\varphi}(\varepsilon))$ and $\widehat{\boldsymbol{\tau}} \coloneqq \nabla\widehat{\varphi}$. The scaled magneto-electro-thermo-elastic state $\mathcal{U}(\varepsilon) \coloneqq (\mathbf{u}(\varepsilon), \bar{\varphi}(\varepsilon), \zeta(\varepsilon), \theta(\varepsilon))$ is the unique solution to the following scaled problem:

$$\left\{ \begin{array}{l} \text{Find } \mathcal{U}(\varepsilon) \in \mathbf{H}^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_\pm) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0), \text{ such that} \\ \text{for all } \mathcal{V} \in \mathbf{H}^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_\pm) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0), \\ A(\varepsilon)(\mathcal{U}(\varepsilon)(t),\mathcal{V}) = L_2(\varepsilon)(\mathcal{V}),\ t \in (0,T), \end{array} \right. \tag{4.17}$$

with initial conditions $(\mathbf{u}_0, \mathbf{u}_1, \theta_0)$, where

$$\begin{aligned} L_2(\varepsilon)(\mathcal{V}) \coloneqq{}& (\mathbf{f},\mathbf{v}) + (\mathbf{g},\mathbf{v})_{\mathbf{L}^2(\widehat{\Gamma})} + (r,\eta) - \tfrac{1}{T_0}(\varrho,\eta)_{L^2(\widehat{\Gamma})} + (\rho_e,\psi) - (b,\xi)_{L^2(\widehat{\Gamma})} + \\ & -a_\varphi(\varepsilon)(\widehat{\varphi},\psi) - b(\varepsilon)(\widehat{\varphi},\mathbf{v}) + d(\varepsilon)(\eta,\partial_t\widehat{\varphi}) - g(\varepsilon)(\xi,\widehat{\varphi}). \end{aligned}$$

3) *The actuator model.* We assume the following scalings for φ^ε and ζ^ε:

$$\begin{aligned} \bar{\varphi}^\varepsilon(x^\varepsilon,t) &= \varepsilon\bar{\varphi}(\varepsilon)(x,t) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T), \\ \bar{\zeta}^\varepsilon(x^\varepsilon,t) &= \varepsilon\bar{\zeta}(\varepsilon)(x,t) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T). \end{aligned} \tag{4.18}$$

Moreover, the components of the scaled vector fields $\boldsymbol{\tau}(\varepsilon) = (\tau_i(\varepsilon))$ and $\boldsymbol{\chi}(\varepsilon) = (\chi_i(\varepsilon))$ are now defined by

$$\begin{aligned} \tau_\alpha(\varepsilon) &\coloneqq \varepsilon\partial_\alpha\varphi(\varepsilon), \quad \tau_3(\varepsilon) \coloneqq \partial_3\varphi(\varepsilon), \\ \chi_\alpha(\varepsilon) &\coloneqq \varepsilon\partial_\alpha\zeta(\varepsilon), \quad \chi_3(\varepsilon) \coloneqq \partial_3\zeta(\varepsilon). \end{aligned}$$

The scaled magneto-electro-thermo-elastic state $\mathcal{U}(\varepsilon) \coloneqq (\mathbf{u}(\varepsilon), \bar{\varphi}(\varepsilon), \bar{\zeta}(\varepsilon), \theta(\varepsilon))$ is the unique solution to the following scaled problem:

$$\left\{ \begin{array}{l} \text{Find } \mathcal{U}(\varepsilon) \in \mathbf{H}^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_\pm) \times H^1(\Omega,\Gamma_\pm) \times H^1(\Omega,\Gamma_0), \text{ such that} \\ \text{for all } \mathcal{V} \in \mathbf{H}^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_\pm) \times H^1(\Omega,\Gamma_\pm) \times H^1(\Omega,\Gamma_0), \\ A(\varepsilon)(\mathcal{U}(\varepsilon)(t),\mathcal{V}) = L_3(\varepsilon)(\mathcal{V}),\ t \in (0,T), \end{array} \right. \tag{4.19}$$

with initial conditions $(\mathbf{u}_0, \mathbf{u}_1, \theta_0)$, where

$$\begin{aligned} L_3(\varepsilon)(\mathcal{V}) \coloneqq{}& (\mathbf{f},\mathbf{v}) + (\mathbf{g},\mathbf{v})_{\mathbf{L}^2(\widehat{\Gamma})} + (r,\eta) - \tfrac{1}{T_0}(\varrho,\eta)_{L^2(\widehat{\Gamma})} + (\rho_e,\psi) + \\ & -a_\zeta(\varepsilon)(\widehat{\zeta},\psi) - f(\varepsilon)(\widehat{\zeta},\mathbf{v}) + e(\varepsilon)(\eta,\partial_t\widehat{\zeta}) - g(\varepsilon)(\widehat{\zeta},\psi) + \\ & -a_\varphi(\varepsilon)(\widehat{\varphi},\psi) - b(\varepsilon)(\widehat{\varphi},\mathbf{v}) + d(\varepsilon)(\eta,\partial_t\widehat{\varphi}) - g(\varepsilon)(\xi,\widehat{\varphi}). \end{aligned}$$

4) *The sensor model.* We assume the following scalings for φ^ε and ζ^ε:

$$\begin{aligned} \varphi^\varepsilon(x^\varepsilon,t) &= \varphi(\varepsilon)(x,t) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T), \\ \zeta^\varepsilon(x^\varepsilon,t) &= \zeta(\varepsilon)(x,t) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\ t \in (0,T). \end{aligned} \tag{4.20}$$

Moreover, the components of the scaled vector fields $\boldsymbol{\tau}(\varepsilon) = (\tau_i(\varepsilon))$ and $\boldsymbol{\chi}(\varepsilon) = (\chi_i(\varepsilon))$ are now defined by

$$\begin{aligned} \tau_\alpha(\varepsilon) &:= \partial_\alpha \varphi(\varepsilon), \quad \tau_3(\varepsilon) := \tfrac{1}{\varepsilon}\partial_3\varphi(\varepsilon), \\ \chi_\alpha(\varepsilon) &:= \partial_\alpha \zeta(\varepsilon), \quad \chi_3(\varepsilon) := \tfrac{1}{\varepsilon}\partial_3\zeta(\varepsilon). \end{aligned}$$

The scaled magneto-electro-thermo-elastic state $\mathcal{U}(\varepsilon) := (\mathbf{u}(\varepsilon), \varphi(\varepsilon), \zeta(\varepsilon), \theta(\varepsilon))$ is the unique solution to the following scaled problem:

$$\left\{ \begin{array}{l} \text{Find } \mathcal{U}(\varepsilon) \in \mathbf{H}^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0), \text{ such that} \\ \text{for all } \mathcal{V} = (\mathbf{v}, \psi, \xi, \eta) \in \mathbf{H}^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0), \\ A(\varepsilon)(\mathcal{U}(\varepsilon)(t), \mathcal{V}) = L_4(\varepsilon)(\mathcal{V}),\ t \in (0, T), \end{array} \right. \tag{4.21}$$

with initial conditions $(\mathbf{u}_0, \mathbf{u}_1, \theta_0)$, where

$$\begin{aligned} L_4(\varepsilon)(\mathcal{V}) := (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{v})_{\mathbf{L}^2(\hat{\Gamma})} + (r, \eta) - \frac{1}{T_0}(\varrho, \eta)_{L^2(\hat{\Gamma})} + \\ + (\rho_e, \psi) - (b, \xi)_{L^2(\hat{\Gamma})} - (d, \psi)_{L^2(\hat{\Gamma})}. \end{aligned}$$

4.1.6 Convergence Results

Preliminarily, we introduce the functional spaces

$$\begin{aligned} \mathbb{X}(\Omega) &:= \{\xi \in L^2(\Omega),\ \partial_3\xi \in L^2(\Omega)\} \equiv H^1(-h, h; L^2(\omega)), \\ \mathbb{X}_0(\Omega) &:= \{\xi \in L^2(\Omega),\ \partial_3\xi \in L^2(\Omega),\ \xi = 0 \text{ on } \Gamma_\pm\}, \end{aligned}$$

usually employed in the asymptotic analysis of actuator piezoelectric plates (see, e.g., [58]). Also, let

$$\mathbf{V}_{KL}(\Omega) := \{\mathbf{v} \in \mathbf{H}^1(\Omega, \Gamma_0);\ e_{i3}(\mathbf{v}) = 0\}$$

denote the space of *Kirchhoff-Love displacements*, and

$$\begin{aligned} \mathbf{V}_H(\omega, \gamma_0) &:= \{\mathbf{v}_H = (v_\alpha) \in \mathbf{H}^1(\omega);\ \mathbf{v}_H = \mathbf{0} \text{ on } \gamma_0\}, \\ V_3(\omega, \gamma_0) &:= \{v_3 \in H^2(\omega);\ v_3 = 0 \text{ and } \partial_\nu v_3 = 0 \text{ on } \gamma_0\}, \end{aligned}$$

where $\boldsymbol{\nu} = (\nu_\alpha)$ is the outer unit normal vector to γ. We recall that $\boldsymbol{\tau} = (-\nu_2, \nu_1)$ represents the unit tangent vector to γ.

As we shall prove in the following subsections, the limit displacement field is always a Kirchhoff-Love field; thus, for consistency reasons, we consider initial conditions such that (see [45])

$$\left\{ \begin{array}{l} \mathbf{u}(\varepsilon)(0) = \mathbf{u}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{V}_{KL}(\Omega), \\ \partial_t \mathbf{u}(\varepsilon)(0) = \mathbf{u}_1 \in \mathbf{V}_{KL}(\Omega), \\ \theta(\varepsilon)(0) = \theta_0 \in H^2(\Omega) \cap H^1(\Omega, \Gamma_0). \end{array} \right. \tag{4.22}$$

4.1.6.1 The Sensor-Actuator model

Theorem 4.3. *Under assumption (4.22), the sequence* $\{\mathcal{U}(\varepsilon)\}_{\varepsilon>0}$ *weakly converges to the limit* $\widetilde{\mathcal{U}} := (\tilde{\mathbf{u}}, \tilde{\varphi}, \tilde{\zeta}, \tilde{\theta})$ *in the space* $L^2(0, T; \mathbf{H}^1(\Omega)) \times L^2(0, T; H^1(\Omega)) \times L^2(0, T; \mathbb{X}(\Omega)) \times L^2(0, T; H^1(\Omega))$.

Proof. For the sake of clarity, the proof is divided into four parts. The first three parts are devoted to showing that the sequence associated with the scaled energy $\{\mathcal{E}(\varepsilon)(t)\}_{\varepsilon>0}$ is uniformly bounded; the proof of the weak convergence result is then accomplished in the fourth part.

(i) *Bounds on* $L^1_{\mathcal{E}}(\varepsilon)$. First we rewrite $(\mathbf{g}, \dot{\mathbf{u}}(\varepsilon))_{\mathbf{L}^2(\hat{\Gamma})} = \partial_t(\mathbf{g}, \mathbf{u}(\varepsilon))_{\mathbf{L}^2(\hat{\Gamma})} - (\dot{\mathbf{g}}, \mathbf{u}(\varepsilon))_{\mathbf{L}^2(\hat{\Gamma})}$ and $f(\varepsilon)(\hat{\zeta}, \dot{\mathbf{u}}(\varepsilon)) = \partial_t[f(\varepsilon)(\hat{\zeta}, \mathbf{u}(\varepsilon))] - f(\varepsilon)(\partial_t\hat{\zeta}, \mathbf{u}(\varepsilon))$. By using, in expression (4.15), Cauchy-Schwarz, Poincaré's, Korn's and Young's inequalities, along with the continuity of the trace operator, we obtain the existence of positive constants C_1 and δ_0 such that

$$L^1_{\mathcal{E}}(\varepsilon)(\mathcal{U}(\varepsilon)) \leqslant C_0(t) + \frac{C_1}{2}\Big\{|\dot{\mathbf{u}}(\varepsilon)|^2_{0,\Omega} + |\boldsymbol{\kappa}(\varepsilon)|^2_{0,\Omega} + |\boldsymbol{\tau}(\varepsilon)|^2_{0,\Omega} + |\theta(\varepsilon)|^2_{0,\Omega} + \\ + |\bar{\chi}(\varepsilon)|^2_{0,\Omega} + \delta_0|\boldsymbol{\gamma}(\varepsilon)|^2\Big\} + \partial_t(\mathbf{g}, \mathbf{u}(\varepsilon))_{\mathbf{L}^2(\hat{\Gamma})} - \partial_t[f(\varepsilon)(\hat{\zeta}, \mathbf{u}(\varepsilon))],$$

where $2C_0(t) := |\mathbf{f}|^2_{0,\Omega} + |\dot{\mathbf{g}}|^2_{0,\hat{\Gamma}} + |\dot{\rho}_e|^2_{0,\Omega} + |\dot{d}|^2_{0,\hat{\Gamma}} + |r|^2_{0,\Omega} + \frac{1}{\delta_0}|\varrho|^2_{0,\hat{\Gamma}} + |\partial_t\hat{\chi}|^2_{0,\Omega}$ depends on the domain and the data (not on ε). Denoting by $K > 0$ the coercivity constant of $a_\theta(\cdot,\cdot)$, with a view toward applying (4.14), we choose δ_0 such that $\widetilde{K} := K - \frac{C_1\delta_0}{2} > 0$. By the definition of the scaled energy and the positive definiteness hypothesis (4.1), there exists a positive constant C_2 such that

$$|\dot{\mathbf{u}}(\varepsilon)|^2_{0,\Omega} + |\boldsymbol{\kappa}(\varepsilon)|^2_{0,\Omega} + |\boldsymbol{\tau}(\varepsilon)|^2_{0,\Omega} + |\theta(\varepsilon)|^2_{0,\Omega} + |\bar{\chi}(\varepsilon)|^2_{0,\Omega} \leqslant C_2\mathcal{E}(\varepsilon),$$

hence there exists $C_3 > 0$ such that

$$L^1_{\mathcal{E}}(\varepsilon) \leqslant C_0(t) + C_3\mathcal{E}(\varepsilon) + \partial_t(\mathbf{g}, \mathbf{u}(\varepsilon))_{\mathbf{L}^2(\hat{\Gamma})} - \partial_t[f(\varepsilon)(\hat{\zeta}, \mathbf{u}(\varepsilon))].$$

(ii) *The sequence* $\{\mathcal{E}(\varepsilon)(0)\}_{\varepsilon>0}$ *is uniformly bounded.* We have:

$$2\mathcal{E}(\varepsilon)(0) = (\rho\mathbf{u}_1, \mathbf{u}_1) + a_u(\varepsilon)(\mathbf{u}_0, \mathbf{u}_0) + a_\varphi(\varepsilon)(\varphi_0(\varepsilon), \varphi_0(\varepsilon)) + a_\zeta(\varepsilon)(\bar{\zeta}_0(\varepsilon), \bar{\zeta}_0(\varepsilon)) + \\ + (c_v\theta_0, \theta_0) - 2d(\varepsilon)(\theta_0, \varphi_0(\varepsilon)) - 2e(\varepsilon)(\theta_0, \bar{\zeta}_0(\varepsilon)) + 2g(\varepsilon)(\bar{\zeta}_0(\varepsilon), \varphi_0(\varepsilon)),$$

where $(\varphi_0(\varepsilon), \bar{\zeta}_0(\varepsilon))$ is the solution to the following variational problem

$$\left\{\begin{array}{l} \text{Find } ((\varphi_0(\varepsilon), \bar{\zeta}_0(\varepsilon)) \in H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_\pm) \\ \text{such that for all } (\psi, \xi) \in H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_\pm), \\ a_\varphi(\varepsilon)(\varphi_0(\varepsilon), \psi) + a_\zeta(\varepsilon)(\bar{\zeta}_0(\varepsilon), \xi) + g(\varepsilon)(\bar{\zeta}_0(\varepsilon), \psi) + g(\varepsilon)(\varphi_0(\varepsilon), \xi) = \\ \quad = (\rho_e(0), \psi) - (d(0), \psi)_{L^2(\hat{\Gamma})} - a_\zeta(\varepsilon)(\hat{\zeta}_0, \xi) + f(\varepsilon)(\xi, \mathbf{u}_0) - e(\varepsilon)(\theta_0, \xi) + \\ \quad\quad - g(\varepsilon)(\hat{\zeta}_0, \psi) + d(\varepsilon)(\theta_0, \psi) + b(\varepsilon)(\psi, \mathbf{u}_0). \end{array}\right. \tag{4.23}$$

By virtue of condition (4.1) (with constant c_1), Cauchy-Schwarz, Poincaré's and Korn's inequalities (with constant c_2) and Young's inequality (with constant δ_1), one has that

$$
\begin{aligned}
& c_1 \left\{ |\boldsymbol{\tau}_0(\varepsilon)|^2_{0,\Omega} + |\bar{\chi}_0(\varepsilon)|^2_{0,\Omega} \right\} \leqslant \\
& \leqslant a_\varphi(\varepsilon)(\varphi_0(\varepsilon), \varphi_0(\varepsilon)) + a_\zeta(\varepsilon)(\bar{\zeta}_0(\varepsilon), \bar{\zeta}_0(\varepsilon)) + 2g(\varepsilon)(\bar{\zeta}_0(\varepsilon), \varphi_0(\varepsilon)) \leqslant \\
& \leqslant c_2 \Big\{ |\boldsymbol{\tau}_0(\varepsilon)|_{0,\Omega}(|\rho_e(0)|_{0,\Omega} + |d(0)|_{0,\hat{\Gamma}} + |\nabla \mathbf{u}_0|_{0,\Omega} + |\theta_0|_{0,\Omega} + |\nabla \widehat{\zeta}_0|_{0,\Omega}) + \\
& \qquad + |\bar{\chi}_0(\varepsilon)|_{0,\Omega}(|\nabla \mathbf{u}_0|_{0,\Omega} + |\theta_0|_{0,\Omega} + |\nabla \widehat{\zeta}_0|_{0,\Omega}) \Big\} \leqslant \\
& \leqslant \frac{c_3 \delta_1}{2} \left\{ |\boldsymbol{\tau}_0(\varepsilon)|^2_{0,\Omega} + |\bar{\chi}_0(\varepsilon)|^2_{0,\Omega} \right\} + \\
& \qquad + \frac{c_3}{2\delta_1} \left\{ |\rho_e(0)|^2_{0,\Omega} + |d(0)|^2_{0,\hat{\Gamma}} + |\nabla \mathbf{u}_0|^2_{0,\Omega} + |\theta_0|^2_{0,\Omega} + |\nabla \widehat{\zeta}_0|^2_{0,\Omega} \right\}.
\end{aligned}
$$

Finally, on choosing δ_1 such that $c_1 - \frac{c_3\delta_1}{2} > 0$, we get the uniform boundedness of $|\boldsymbol{\tau}_0(\varepsilon)|_{0,\Omega}$ and $|\bar{\chi}_0(\varepsilon)|_{0,\Omega}$. Now, we remark that $e_{i3}(\mathbf{u}_0) = 0$ by hypothesis (4.22), which implies

$$
a_u(\varepsilon)(\mathbf{u}_0, \mathbf{u}_0) = \int_\Omega C_{ijk\ell} \kappa_{k\ell}(\varepsilon)(\mathbf{u}_0) \kappa_{ij}(\varepsilon)(\mathbf{u}_0) dx = \int_\Omega C_{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}(\varepsilon)(\mathbf{u}_0) \kappa_{\alpha\beta}(\varepsilon)(\mathbf{u}_0) dx,
$$

so that, by the properties of $\mathbf{C}$, we have

$$
a_u(\varepsilon)(\mathbf{u}_0, \mathbf{u}_0) \leqslant c\, |\nabla \mathbf{u}_0|^2_{0,\Omega}
$$

for some constant $c > 0$. Consequently, by virtue of the above bounds, there exists a constant $c_4 > 0$ such that $\mathcal{E}(\varepsilon)(0) \leqslant c_4$, i.e., the sequence $\{\mathcal{E}(\varepsilon)(0)\}_{\varepsilon>0}$ is uniformly bounded.

(iii) *The sequence* $\{\mathcal{E}(\varepsilon)(t)\}_{\varepsilon>0}$ *is uniformly bounded.* Upon integrating the energy evolution equation (4.14) in $(0, t)$, using the results of steps (i) and (ii), the continuity of the trace operator and Korn's inequality, we infer that

$$
\begin{aligned}
\mathcal{E}(\varepsilon)(t) + \widetilde{K} \int_0^t |\boldsymbol{\gamma}(\varepsilon)(s)|^2_{0,\Omega}\, ds \leqslant \mathcal{E}(\varepsilon)(0) + \int_0^t L^1_{\mathcal{E}}(\varepsilon)(s) ds \leqslant \\
\leqslant c_4 + \int_0^t \Big(C_0(s) + C_3 \mathcal{E}(\varepsilon)(s) + \\
+ \partial_s(\mathbf{g}(s), \mathbf{u}(\varepsilon)(s))_{\mathbf{L}^2(\hat{\Gamma})} - \partial_s[f(\varepsilon)(\widehat{\zeta}(s), \mathbf{u}(\varepsilon)(s))] \Big) ds \leqslant \\
\leqslant C_4(t) + \frac{1}{2} \left(\frac{c_5}{\delta_2} |\widehat{\chi}(t)|^2_{0,\Omega} + \frac{c_6}{\delta_3} |\mathbf{g}(t)|^2_{0,\hat{\Gamma}} \right) + \\
+ \frac{1}{2}(c_5\delta_2 + c_6\delta_3) |\boldsymbol{\kappa}(\varepsilon)(t)|^2_{0,\Omega} + C_3 \int_0^t \mathcal{E}(\varepsilon)(s) ds.
\end{aligned}
$$

Since $|\boldsymbol{\kappa}(\varepsilon)(t)|^2_{0,\Omega}$ is bounded (up to a constant) by $\mathcal{E}(\varepsilon)(t)$, it is sufficient to select suitable values of δ_2 and δ_3 to get the following estimate:

$$
\mathcal{E}(\varepsilon)(t) + \widetilde{K} \int_0^t |\boldsymbol{\gamma}(\varepsilon)(s)|^2_{0,\Omega}\, ds \leqslant C_5(t) + C_3 \int_0^t \mathcal{E}(\varepsilon)(s) ds \leqslant C_6 + C_3 \int_0^t \mathcal{E}(\varepsilon)(s) ds,
$$

with $C_6 \coloneqq \sup_{t\in(0,T)} C_5(t)$. Thanks to Gronwall's inequality, there exist two positive constants m and k such that

$$\mathcal{E}(\varepsilon)(t) \leqslant me^{kt} \text{ and } \widetilde{K} \int_0^t |\boldsymbol{\gamma}(\varepsilon)(s)|^2_{0,\Omega}\, ds \leqslant me^{kt} \text{ for all } t \in (0,T).$$

(iv) *Weak convergences.* We are now in a position to establish the weak convergence result. From the bound on the energy we infer that the sequences $\{\boldsymbol{\kappa}(\varepsilon)\}_{\varepsilon>0}$, $\{\bar{\boldsymbol{\chi}}(\varepsilon)\}_{\varepsilon>0}$, $\{\boldsymbol{\tau}(\varepsilon)\}_{\varepsilon>0}$ and $\{\boldsymbol{\gamma}(\varepsilon)\}_{\varepsilon>0}$ are uniformly bounded in $L^2(0,T;\mathbf{L}^2(\Omega))$, therefore we have the following weak convergences (up to a subsequence):

$$\begin{array}{ll} \kappa_{ij}(\varepsilon) \rightharpoonup \tilde{\kappa}_{ij} & \text{in } L^2(0,T;L^2(\Omega)),\\ \bar{\chi}_i(\varepsilon) \rightharpoonup \tilde{\bar{\chi}}_i & \text{in } L^2(0,T;L^2(\Omega)),\\ \tau_i(\varepsilon) \rightharpoonup \tilde{\tau}_i & \text{in } L^2(0,T;L^2(\Omega)),\\ \gamma_i(\varepsilon) \rightharpoonup \tilde{\gamma}_i & \text{in } L^2(0,T;L^2(\Omega)). \end{array}$$

Moreover, by means of Korn's and Poincaré's inequalities and from the definition of $\kappa_{ij}(\varepsilon)$, $\tau_i(\varepsilon)$ and $\gamma_i(\varepsilon)$, we infer that $\|\mathbf{u}(\varepsilon)\|_{1,\Omega}$, $\|\varphi(\varepsilon)\|_{1,\Omega}$ and $\|\theta(\varepsilon)\|_{1,\Omega}$ are also bounded, so that

$$\begin{array}{ll} \mathbf{u}(\varepsilon) \rightharpoonup \tilde{\mathbf{u}} & \text{in } L^2(0,T;\mathbf{H}^1(\Omega)),\\ \dot{\mathbf{u}}(\varepsilon) \rightharpoonup \dot{\tilde{\mathbf{u}}} & \text{in } L^2(0,T;\mathbf{L}^2(\Omega)),\\ \varphi(\varepsilon) \rightharpoonup \tilde{\varphi} & \text{in } L^2(0,T;H^1(\Omega)),\\ \theta(\varepsilon) \rightharpoonup \tilde{\theta} & \text{in } L^2(0,T;H^1(\Omega)). \end{array}$$

Upon writing

$$\bar{\zeta}(\varepsilon)(\tilde{x},x_3) = \int_{-h}^{x_3} \partial_3\bar{\zeta}(\varepsilon)(\tilde{x},y_3)dy_3,$$

it follows that $|\bar{\zeta}(\varepsilon)|_{0,\Omega} \leqslant 2h|\partial_3\bar{\zeta}(\varepsilon)|_{0,\Omega} \leqslant ce^{mT}$, by virtue of the boundedness of $\bar{\chi}_i(\varepsilon)$. This implies that both $\bar{\zeta}(\varepsilon)$ and $\zeta(\varepsilon)$ are bounded in $L^2(\Omega)$ and thus,

$$\begin{array}{ll} \bar{\zeta}(\varepsilon) \rightharpoonup \tilde{\bar{\zeta}} & \text{in } L^2(0,T;\mathbb{X}_0(\Omega)),\\ \zeta(\varepsilon) \rightharpoonup \tilde{\zeta} & \text{in } L^2(0,T;\mathbb{X}(\Omega)). \end{array}$$

This completes the proof. □

Theorem 4.4. *The weak limit* $\widetilde{\mathcal{U}}(t) = (\tilde{\mathbf{u}}(t),\tilde{\varphi}(t),\tilde{\zeta}(t),\tilde{\theta}(t))$ *is the solution to the limit variational problem:*

$$\left\{\begin{array}{l} \textit{Find } \widetilde{\mathcal{U}}(t) \in \mathbf{V}_{KL}(\Omega)\times H^1(\Omega,\Gamma_0)\times\mathbb{X}(\Omega)\times H^1(\Omega,\Gamma_0),\ t\in(0,T)\ \textit{such that}\\ \widetilde{A}_1(\widetilde{\mathcal{U}}(t),\mathcal{V}) = \widetilde{L}_1(\mathcal{V}),\ \textit{for all } \mathcal{V}\in\mathbf{V}_{KL}(\Omega)\times H^1(\Omega,\Gamma_0)\times\mathbb{X}_0(\Omega)\times H^1(\Omega,\Gamma_0),\\ \tilde{\zeta} = \zeta^{\pm} \textit{ on } \Gamma_{\pm}, \end{array}\right. \tag{4.24}$$

where

$$\widetilde{A}_1(\widetilde{\mathcal{U}}(t),\mathcal{V}) := \int_\Omega \Big\{ \Big(\widetilde{C}^1_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{P}^1_{\sigma\alpha\beta}\partial_\sigma\tilde{\varphi}(t) + \widetilde{R}^1_{3\alpha\beta}\partial_3\tilde{\zeta}(t) - \widetilde{\beta}^1_{\alpha\beta}\tilde{\theta}(t)\Big) e_{\alpha\beta}(\mathbf{v}) +$$
$$+ \Big(-\widetilde{P}^1_{\alpha\sigma\tau} e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{X}^1_{\alpha\beta}\partial_\beta\tilde{\varphi}(t) + \widetilde{\alpha}^1_{\alpha 3}\partial_3\tilde{\zeta}(t) - \widetilde{p}^1_\alpha\tilde{\theta}(t)\Big)\partial_\alpha\psi +$$
$$+ \Big(-\widetilde{R}^1_{3\alpha\beta} e_{\alpha\beta}(\tilde{\mathbf{u}}(t)) + \widetilde{\alpha}^1_{\alpha 3}\partial_\alpha\tilde{\varphi}(t) + \widetilde{M}^1_{33}\partial_3\tilde{\zeta}(t) - \widetilde{m}^1_3\tilde{\theta}(t)\Big)\partial_3\xi +$$
$$+ \Big(\widetilde{\beta}^1_{\alpha\beta} e_{\alpha\beta}(\dot{\tilde{\mathbf{u}}}(t)) - \widetilde{m}^1_3\partial_3\dot{\tilde{\zeta}}(t) - \widetilde{p}^1_\alpha\partial_\alpha\dot{\tilde{\varphi}}(t) + \widetilde{c}^1_v\dot{\tilde{\theta}}(t)\Big)\eta +$$
$$+ \widetilde{K}^1_{\alpha\beta}\partial_\beta\tilde{\theta}(t)\partial_\alpha\eta + \rho\ddot{\tilde{u}}_i(t)v_i \Big\}\, dx,$$
$$\widetilde{L}_1(\mathcal{V}) := (\mathbf{f},\mathbf{v}) + (\mathbf{g},\mathbf{v})_{\mathbf{L}^2(\widehat{\Gamma})} + (r,\eta) - \tfrac{1}{T_0}(\varrho,\eta)_{L^2(\widehat{\Gamma})} + (\rho_e,\psi) - (d,\psi)_{L^2(\widehat{\Gamma})}.$$

The reduced magneto-electro-thermo-elastic coefficients $\widetilde{C}^1_{\alpha\beta\sigma\tau}$, $\widetilde{X}^1_{\alpha\beta}$, $\widetilde{K}^1_{\alpha\beta}$, $\widetilde{P}^1_{\sigma\alpha\beta}$, $\widetilde{\beta}^1_{\alpha\beta}$, $\widetilde{p}^1_\alpha$ $\widetilde{m}^1_3$, $\widetilde{R}^1_{3\alpha\beta}$ *and* $\widetilde{\alpha}^1_{\alpha 3}$ *are listed in Appendix 1.*

Proof. For the sake of clarity the proof is split into three parts.
(i) By the definition of $\kappa_{ij}(\varepsilon)$, $\chi_i(\varepsilon)$, $\tau_i(\varepsilon)$ and $\gamma_i(\varepsilon)$, and thanks to the results of Theorem 4.3, there exists two constants C_M and C_K such that

$$\begin{array}{lll} |e_{\alpha\beta}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant C_M e^{C_K T}, & |e_{\alpha 3}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}, & |e_{33}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant \varepsilon^2 C_M e^{C_K T}, \\ |\partial_\alpha\zeta(\varepsilon)|_{0,\Omega} \leqslant \frac{1}{\varepsilon} C_M e^{C_K T}, & |\partial_3\zeta(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T}, & \\ |\partial_\alpha\varphi(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T}, & |\partial_3\varphi(\varepsilon)|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}, & \\ |\partial_\alpha\theta(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T}, & |\partial_3\theta(\varepsilon)|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}. & \end{array} \tag{4.25}$$

From the first set of inequalities $(4.25)_1$, we get that $e_{i3}(\mathbf{u}(\varepsilon)(t)) \to 0$ in $L^2(\Omega)$ for almost every $t \in (0,T)$. Also, as $\mathbf{u}(\varepsilon)(t) \rightharpoonup \tilde{\mathbf{u}}(t)$ in $\mathbf{H}^1(\Omega)$, we have that $e_{i3}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup e_{i3}(\tilde{\mathbf{u}}(t))$ and so $e_{i3}(\tilde{\mathbf{u}}(t)) = 0$ by uniqueness of the limit. This implies that $\partial_3\tilde{u}_3 = 0$, i.e., $\tilde{u}_3(\tilde{x}, x_3) = \tilde{u}_3(\tilde{x})$ is independent of x_3. We also have that $\partial_3\tilde{u}_\alpha = -\partial_\alpha\tilde{u}_3$, i.e., $\tilde{u}_\alpha(\tilde{x}, x_3) = \tilde{u}_\alpha(\tilde{x}) - x_3\partial_\alpha\tilde{u}_3(\tilde{x})$. Consequently, $\tilde{\mathbf{u}}(t) \in \mathbf{V}_{KL}(\Omega)$. Moreover, we obtain that $e_{\alpha\beta}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{\alpha\beta}(t) = e_{\alpha\beta}(\tilde{\mathbf{u}}(t))$ in $L^2(\Omega)$, $\frac{1}{\varepsilon}e_{\alpha 3}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{\alpha 3}(t)$ in $L^2(\Omega)$, $\frac{1}{\varepsilon}e_{33}(\mathbf{u}(\varepsilon)(t)) \to 0$ in $L^2(\Omega)$ and, also, $\frac{1}{\varepsilon^2}e_{33}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{33}(t)$ in $L^2(\Omega)$.

From the second set of inequalities $(4.25)_2$, we have that $\varepsilon\partial_\alpha\zeta(\varepsilon)(t) \to 0$ in $L^2(\Omega)$ and $\partial_3\zeta(\varepsilon)(t) \rightharpoonup \partial_3\tilde{\zeta}(t)$ in $L^2(\Omega)$.

From the last sets of inequalities $(4.25)_{3,4}$, since $\varphi(\varepsilon)(t) \rightharpoonup \tilde{\varphi}(t)$ in $H^1(\Omega)$, we infer that $\partial_\alpha\varphi(\varepsilon)(t) \rightharpoonup \tilde{\tau}_\alpha(t) = \partial_\alpha\tilde{\varphi}(t)$ in $L^2(\Omega)$ and, also, $\partial_3\varphi(\varepsilon)(t) \to 0$ in $L^2(\Omega)$, i.e., $\tilde{\varphi}(t) = \tilde{\varphi}(\tilde{x})(t)$ is independent of x_3. Besides, $\frac{1}{\varepsilon}\partial_3\varphi(\varepsilon)(t) \rightharpoonup \tau_3(t)$ in $L^2(\Omega)$. Similarly, since $\theta(\varepsilon)(t) \rightharpoonup \tilde{\theta}(t)$ in $H^1(\Omega)$, we obtain that $\partial_\alpha\theta(\varepsilon)(t) \rightharpoonup \tilde{\gamma}_\alpha(t) = \partial_\alpha\tilde{\theta}(t)$ in $L^2(\Omega)$, $\partial_3\theta(\varepsilon)(t) \to 0$ in $L^2(\Omega)$, i.e., $\tilde{\theta}(t) = \tilde{\theta}(\tilde{x})(t)$ and, finally, $\frac{1}{\varepsilon}\partial_3\theta(\varepsilon)(t) \rightharpoonup \gamma_3(t)$ in $L^2(\Omega)$.
(ii) *Computations of* κ_{i3}, τ_3 *and* γ_3. Let us multiply problem (4.13) by ε^2 and let ε tend to zero. We get the following equation:

$$C_{3333}\kappa_{33} + 2C_{\alpha 333}\kappa_{\alpha 3} + P_{333}\tau_3 + C_{\alpha\beta 33}e_{\alpha\beta}(\tilde{\mathbf{u}}) + P_{\alpha 33}\partial_\alpha\tilde{\varphi} + R_{333}\partial_3\tilde{\zeta} - \beta_{33}\tilde{\theta} = 0.$$

By multiplying problem (4.13) by ε, choosing test functions $v_3 = \psi = \eta = \xi = 0$ and letting ε tend to zero, we have that

$$C_{\alpha 333}\kappa_{33} + 2C_{\alpha 3\beta 3}\kappa_{\beta 3} + P_{3\alpha 3}\tau_3 + C_{\sigma\beta\alpha 3}e_{\sigma\beta}(\tilde{\mathbf{u}}) + P_{\sigma\alpha 3}\partial_\sigma\tilde{\varphi} + R_{3\alpha 3}\partial_3\tilde{\zeta} - \beta_{\alpha 3}\tilde{\theta} = 0.$$

Similarly, by multiplying problem (4.13) by ε and choosing test functions $v_i = \eta = \xi = 0$, when ε tends to zero, we find

$$-P_{333}\kappa_{33} - 2P_{3\alpha 3}\kappa_{\alpha 3} + X_{33}\tau_3 - P_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}) + X_{\alpha 3}\partial_\alpha\tilde{\varphi} + \alpha_{33}\partial_3\tilde{\zeta} - p_3\tilde{\theta} = 0.$$

Finally, if we multiply by ε and choose test functions $v_i = \psi = \xi = 0$, we obtain the last equation

$$K_{33}\gamma_3 + K_{\alpha 3}\partial_\alpha\tilde{\theta} = 0.$$

By combining the whole set of equations above we are now in a position to characterize κ_{i3}, τ_3 and γ_3. Let $\mathbf{l}^1 = (l_i^1)$ be the vector whose components are defined by $l_\alpha^1 \coloneqq 2\kappa_{\alpha 3}$ and $l_3^1 \coloneqq \kappa_{33}$, and $(d_{ij}) \coloneqq (C_{i3j3})^{-1}$, then

$$\begin{aligned}
l_i^1 &= -d_{ij}\Big\{\Big(C_{\alpha\beta j3} + k'P_{3j3}P'_{3\alpha\beta}\Big)e_{\alpha\beta}(\tilde{\mathbf{u}}) + \big(P_{\alpha j3} - k'P_{3j3}X'_{\alpha 3}\big)\partial_\alpha\tilde{\varphi} + \\
&\quad + \big(R_{3j3} - k'P_{3j3}\alpha'_{33}\big)\partial_3\tilde{\zeta} - \big(\beta_{j3} - k'P_{3j3}p'_3\big)\tilde{\theta}\Big\}, \\
\tau_3 &= k'\Big(P'_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}) - X'_{\alpha 3}\partial_\alpha\tilde{\varphi} - \alpha'_{33}\partial_3\tilde{\zeta} + p'_3\tilde{\theta}\Big), \\
\gamma_3 &= -K'_{\alpha 3}\partial_\alpha\tilde{\theta}.
\end{aligned} \tag{4.26}$$

Coefficients k', $P'_{3\alpha\beta}$, $X'_{\alpha 3}$, α'_{33}, p'_3 and $K'_{\alpha 3}$ are defined in Appendix 1.
(iii) *Definition of the limit problem.* We let test functions be $\mathcal{V} = (\mathbf{v}, \psi, \xi, \eta) \in \mathbf{V}_{KL}(\Omega) \times H^1(\Omega, \Gamma_0) \times \mathbb{X}_0(\Omega) \times H^1(\Omega, \Gamma_0)$ in problem (4.13) and let $\varepsilon \to 0$, by substituting expressions (4.26), we obtain, as customary, the limit evolution problem

$$\left\{\begin{aligned}
&\text{Find } \widetilde{\mathcal{U}}(t) \in \mathbf{V}_{KL}(\Omega) \times H^1(\Omega, \Gamma_0) \times \mathbb{X}(\Omega) \times H^1(\Omega, \Gamma_0),\ t \in (0, T) \text{ such that} \\
&\int_\Omega \Big\{\Big(\widetilde{C}^1_{\alpha\beta\sigma\tau}e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{P}^1_{\sigma\alpha\beta}\partial_\sigma\tilde{\varphi}(t) + \widetilde{R}^1_{3\alpha\beta}\partial_3\tilde{\zeta}(t) - \widetilde{\beta}^1_{\alpha\beta}\tilde{\theta}(t)\Big)e_{\alpha\beta}(\mathbf{v}) + \\
&\quad + \Big(-\widetilde{P}^1_{\alpha\sigma\tau}e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{X}^1_{\alpha\beta}\partial_\beta\tilde{\varphi}(t) + \widetilde{\alpha}^1_{\alpha 3}\partial_3\tilde{\zeta}(t) - \widetilde{p}^1_\alpha\tilde{\theta}(t)\Big)\partial_\alpha\psi + \\
&\quad + \Big(-\widetilde{R}^1_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}(t)) + \widetilde{\alpha}^1_{\alpha 3}\partial_\alpha\tilde{\varphi}(t) + \widetilde{M}^1_{33}\partial_3\tilde{\zeta}(t) - \widetilde{m}^1_3\tilde{\theta}(t)\Big)\partial_3\xi + \\
&\quad + \Big(\widetilde{\beta}^1_{\alpha\beta}e_{\alpha\beta}(\dot{\tilde{\mathbf{u}}}(t)) - \widetilde{m}^1_3\partial_3\dot{\tilde{\zeta}}(t) - \widetilde{p}^1_\alpha\partial_\alpha\dot{\tilde{\varphi}}(t) + \widetilde{c}^1_v\dot{\tilde{\theta}}(t)\Big)\eta + \\
&\quad + \widetilde{K}^1_{\alpha\beta}\partial_\beta\tilde{\theta}(t)\partial_\alpha\eta + \rho\ddot{\tilde{u}}_i(t)v_i\Big\}\,dx = \widetilde{L}_1(\mathcal{V}), \\
&\tilde{\zeta} = \zeta^\pm \text{ on } \Gamma_\pm,\ \text{for all } \mathcal{V} \in \mathbf{V}_{KL}(\Omega) \times H^1(\Omega, \Gamma_0) \times \mathbb{X}_0(\Omega) \times H^1(\Omega, \Gamma_0)
\end{aligned}\right. \tag{4.27}$$

Problem (4.27) is formally equivalent to (4.24), presented in the statement of Theorem 4.4.

□

The limit evolution problem

In this section we present the variational and differential formulations of the evolution problem for a sensor-actuator magneto-electro-thermo-elastic plate. The primary unknowns of the limit problem are collected into the limit magneto-electro-thermo-elastic state $\widetilde{\mathcal{U}}(t) = (\tilde{\mathbf{u}}(t), \tilde{\varphi}(t), \tilde{\zeta}(t), \tilde{\theta}(t)) \in \mathbf{V}_{KL}(\Omega) \times H^1(\Omega, \Gamma_0) \times \mathbb{X}(\Omega) \times H^1(\Omega, \Gamma_0)$, whose components take the following form:

$$\begin{aligned}&\tilde{u}_\alpha(\tilde{x},x_3)=u_\alpha(\tilde{x})-x_3\partial_\alpha u_3(\tilde{x}),\ \ \mathbf{u}_H:=(u_\alpha),\\&\tilde{u}_3(\tilde{x},x_3)=u_3(\tilde{x}),\\&\tilde{\varphi}(\tilde{x},x_3)=\phi(\tilde{x}),\\&\tilde{\theta}(\tilde{x},x_3)=\vartheta(\tilde{x}).\end{aligned}\tag{4.28}$$

In the case of a homogeneous material, the reduced magneto-electro-thermo-elastic coefficients $\widetilde{C}^1_{\alpha\beta\sigma\tau}$, $\widetilde{X}^1_{\alpha\beta}$, $\widetilde{K}^1_{\alpha\beta}$, $\widetilde{P}^1_{\sigma\alpha\beta}$, $\widetilde{\beta}^1_{\alpha\beta}$, $\widetilde{p}^1_\alpha$ $\widetilde{m}^1_3$, $\widetilde{R}^1_{3\alpha\beta}$ and $\widetilde{\alpha}^1_{\alpha3}$ are constant functions and, thus, the limit evolution problem decouples into two variational subproblems, namely, the flexural problem, which gives u_3, and the thermo-piezoelectric membrane problem, which gives the triplet $(\mathbf{u}_H,\phi,\vartheta)$. Moreover, we can characterize explicitly the limit magnetic potential $\tilde{\zeta}$ as a second order polynomial function of x_3.

Indeed, by choosing test functions $\mathcal{V}=(\mathbf{0},0,\xi,0)$ in (4.27), after an integration by parts along x_3, we get the expression of the limit magnetic potential $\tilde{\zeta}$:

$$\tilde{\zeta}(\tilde{x},x_3)=\sum_{k=0}^{2}z^k(\tilde{x})x_3^k,\tag{4.29}$$

where

$$z^0=\langle\zeta\rangle+\frac{h^2\widetilde{R}^1_{3\alpha\beta}}{2\widetilde{M}^1_{33}}\partial_{\alpha\beta}u_3,\ \ z^1=\frac{[\![\zeta]\!]}{2h},\ \ z^2=-\frac{\widetilde{R}^1_{3\alpha\beta}}{2\widetilde{M}^1_{33}}\partial_{\alpha\beta}u_3.$$

with $\langle\zeta\rangle:=\frac{\zeta^++\zeta^-}{2}$ and $[\![\zeta]\!]:=\zeta^+-\zeta^-$ representing, respectively, the mean value and the jump function between the top and bottom boundary values of ζ. As the careful reader can notice, the limit magnetic potential becomes a known function depending on the transversal displacement u_3 of the plate and on the values of the magnetic potentials $\zeta^\pm$, applied on the upper and lower surfaces $\Gamma_\pm$.

By virtue of the characterization (4.29) of the limit magnetic potential $\tilde{\zeta}$, we can now rewrite the limit evolution problem. After an integration by parts along x_3, we obtain a two-dimensional problem defined over the middle surface ω of the plate:

$$\left\{\begin{aligned}&\text{Find }\widetilde{\mathcal{U}}=(\mathbf{u}_H(t),u_3(t),\phi(t),\vartheta(t))\in\mathbf{V}_H(\omega,\gamma_0)\times V_3(\omega,\gamma_0)\times\\&\times H^1(\omega,\gamma_0)\times H^1(\omega,\gamma_0)\text{ such that}\\&2h\int_\omega\left\{\mathcal{N}^1_{\alpha\beta}(\mathbf{u}_H(t),\phi(t),\vartheta(t))e_{\alpha\beta}(\mathbf{v}_H)+\rho\ddot{u}_\alpha(t)v_\alpha\right\}d\tilde{x}+\\&2h\int_\omega\left\{-\widetilde{D}^1_\alpha(\mathbf{u}_H(t),\phi(t),\vartheta(t))\partial_\alpha\psi+\partial_t\widetilde{\mathcal{S}}^1(\mathbf{u}_H(t),\phi(t),\vartheta(t))\eta-\widetilde{q}^1_\alpha(\vartheta(t))\partial_\alpha\eta\right\}d\tilde{x}\\&\quad+\frac{2h^3}{3}\int_\omega\left\{\mathcal{M}^1_{\alpha\beta}(u_3(t))\partial_{\alpha\beta}v_3+\rho\partial_\alpha\ddot{w}(t)\partial_\alpha v_3+\frac{3}{h^2}\rho\ddot{w}(t)v_3\right\}d\tilde{x}=\mathcal{L}_1(\widetilde{\mathcal{V}}),\\&\text{for all }\widetilde{\mathcal{V}}=(\mathbf{v}_H,v_3,\psi,\eta)\in\mathbf{V}_H(\omega,\gamma_0)\times V_3(\omega,\gamma_0)\times H^1(\omega,\gamma_0)\times H^1(\omega,\gamma_0),\end{aligned}\right.$$

with

$$\begin{aligned}\mathcal{L}_1(\widetilde{\mathcal{V}}):=&\int_\omega\left\{\widetilde{f}_iv_i-m_\alpha\partial_\alpha v_3+\widetilde{\rho}_e\psi+\widetilde{r}\eta\right\}d\tilde{x}+\int_{\gamma_1}\left\{\widetilde{g}_iv_i-n_\alpha\partial_\alpha v_3-\widetilde{d}\psi-\widetilde{\varrho}\eta\right\}d\gamma\\&-\int_\omega\left\{\widetilde{R}^1_{3\alpha\beta}[\![\zeta]\!]e_{\alpha\beta}(\mathbf{v}_H)+\widetilde{\alpha}^1_{\alpha3}[\![\zeta]\!]\partial_\alpha\psi-\widetilde{m}^1_3[\![\dot{\zeta}]\!]\eta\right\}d\tilde{x}.\end{aligned}$$

The initial conditions are given by

$$\begin{cases} \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0 = (u_{\alpha,0} - x_3\partial_\alpha u_{3,0}, u_{3,0}), \\ \dot{\tilde{\mathbf{u}}}(0) = \tilde{\mathbf{u}}_1 = (u_{\alpha,1} - x_3\partial_\alpha u_{3,1}, u_{3,1}), \\ \tilde{\theta}(0) = \tilde{\theta}_0 = \vartheta_0. \end{cases}$$

$(\mathcal{N}^1_{\alpha\beta})$, $(\mathcal{M}^1_{\alpha\beta})$, $(\widetilde{D}^1_\alpha)$, $\widetilde{\mathcal{S}}^1$ and $(\widetilde{q}^1_\alpha)$ represent, respectively, the membrane stress tensor, the moment tensor, the reduced electric displacement vector, the reduced thermodynamic entropy and the reduced heat flow vector of the plate, whose components are defined by the constitutive laws below:

$$\begin{cases} \mathcal{N}^1_{\alpha\beta} := \widetilde{C}^1_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}_H) + \widetilde{P}^1_{\sigma\alpha\beta}\partial_\sigma\phi - \widetilde{\beta}^1_{\alpha\beta}\vartheta, \\ \widetilde{D}^1_\alpha := \widetilde{P}^1_{\alpha\sigma\tau} e_{\sigma\tau}(\mathbf{u}_H) - \widetilde{X}^1_{\alpha\beta}\partial_\beta\phi + \widetilde{p}^1_\alpha\vartheta, \\ \widetilde{\mathcal{S}}^1 := \widetilde{\beta}^1_{\alpha\beta} e_{\alpha\beta}(\mathbf{u}_H) - \widetilde{p}^1_\alpha\partial_\alpha\phi + \widetilde{c}^1_v\vartheta, \\ \mathcal{M}^1_{\alpha\beta} := \widetilde{A}^1_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}u_3, \\ \widetilde{q}^1_\alpha := -\widetilde{K}^1_{\alpha\beta}\partial_\beta\vartheta, \end{cases}$$

where $\widetilde{A}^1_{\alpha\beta\sigma\tau} := \widetilde{C}^1_{\alpha\beta\sigma\tau} + \frac{\widetilde{R}^1_{3\alpha\beta}\widetilde{R}^1_{3\sigma\tau}}{\widetilde{M}^1_{33}}$.

Moreover, the two-dimensional applied thermo-electro-mechanical loads are

$$\begin{aligned} &\widetilde{f}_i(t) := \int_{-h}^{h} f_i(t)dx_3 + g_i^+(t) + g_i^-(t), \\ &m_\alpha(t) := \int_{-h}^{h} x_3 f_\alpha(t)dx_3 + h(g_\alpha^+(t) - g_\alpha^-(t)), \\ &\widetilde{g}_i(t) := \int_{-h}^{h} g_i(t)dx_3, \;\; n_\alpha(t) := \int_{-h}^{h} x_3 g_\alpha(t)dx_3, \\ &\widetilde{\rho}_e(t) := \int_{-h}^{h} \rho_e(t)dx_3 - d^+(t) - d^-(t), \;\; \widetilde{d}(t) := \int_{-h}^{h} d(t)dx_3, \\ &\widetilde{r}(t) := \int_{-h}^{h} r(t)dx_3 - \varrho^+(t) - \varrho^-(t), \;\; \widetilde{\varrho}(t) := \frac{1}{T_0}\int_{-h}^{h} \varrho(t)dx_3, \end{aligned}$$

where $v^\pm := v|_{\Gamma_\pm} = v(\tilde{x}, \pm h)$ denotes the restriction of v to $\Gamma_\pm$.

The variational problem above can be split into two two-dimensional decoupled problems: namely, the flexural problem and the thermo-piezoelectric membrane problem. The flexural problem reads as follows:

$$\begin{cases} \text{Find } u_3(t) \in V_3(\omega,\gamma_0),\; t \in (0,T) \text{ such that} \\ \dfrac{2h^3}{3}\displaystyle\int_\omega \left\{ \mathcal{M}^1_{\alpha\beta}(u_3(t))\partial_{\alpha\beta}v_3 + \rho\partial_\alpha\ddot{u}_3(t)\partial_\alpha v_3 + \frac{3}{h^2}\rho\ddot{u}_3(t)v_3 \right\} d\tilde{x} = \\ \quad = \displaystyle\int_\omega \left\{ \widetilde{f}_3 v_3 - m_\alpha\partial_\alpha v_3 \right\} d\tilde{x} + \int_{\gamma_1} \left\{ \widetilde{g}_3 v_3 - n_\alpha\partial_\alpha v_3 \right\} d\gamma, \\ \text{for all } v_3 \in V_3(\omega,\gamma_0). \end{cases}$$

The two-dimensional thermo-piezoelectric membrane problem takes the following form

$$
\left\{
\begin{array}{l}
\text{Find } (\mathbf{u}_H(t), \phi(t), \vartheta(t)), \in \mathbf{V}_H(\omega,\gamma_0) \times H^1(\omega,\gamma_0) \times H^1(\omega,\gamma_0),\ t \in (0,T) \\
\text{such that } 2h \displaystyle\int_\omega \left\{ \mathcal{N}^1_{\alpha\beta}(\mathbf{u}_H(t), \phi(t), \vartheta(t)) e_{\alpha\beta}(\mathbf{v}_H) + \rho \ddot{u}_\alpha(t) v_\alpha \right\} d\tilde{x} + \\
+2h \displaystyle\int_\omega \left\{ -\widetilde{D}^1_\alpha(\mathbf{u}_H(t), \phi(t), \vartheta(t)) \partial_\alpha \psi + \partial_t \widetilde{\mathcal{S}}^1(\mathbf{u}_H(t), \phi(t), \vartheta(t)) \eta - \widetilde{q}^1_\alpha(\vartheta(t)) \partial_\alpha \eta \right\} d\tilde{x} \\
\quad = \displaystyle\int_\omega \left\{ \widetilde{f}_\alpha v_\alpha + \widetilde{\rho}_e \psi - \widetilde{R}_{3\alpha\beta} [\![\zeta]\!] e_{\alpha\beta}(\mathbf{v}_H) - \widetilde{\alpha}_{\alpha 3} [\![\zeta]\!] \partial_\alpha \psi + (\widetilde{r} + \widetilde{m}_3 [\![\dot{\zeta}]\!]) \eta \right\} d\tilde{x} + \\
\quad + \displaystyle\int_{\gamma_1} \left\{ \widetilde{g}_\alpha v_\alpha - \widetilde{d} \psi - \widetilde{\varrho} \eta \right\} d\gamma, \\
\text{for all } (\mathbf{v}_H, \psi, \eta) \in \mathbf{V}_H(\omega,\gamma_0) \times H^1(\omega,\gamma_0) \times H^1(\omega,\gamma_0).
\end{array}
\right.
$$

We are now in a position to write the decoupled flexural and thermo-piezoelectric membrane problems into their differential form by using Green's formulae on ω.

The transversal displacement u_3 solves the following flexural differential problem:

$$
\left\{
\begin{array}{ll}
\textit{Field equation:} & \\
\frac{2h^3}{3} \partial_{\alpha\beta} \mathcal{M}^1_{\alpha\beta} - \frac{2h^3}{3} \rho \partial_{\alpha\alpha} \ddot{u}_3 + 2h\rho \ddot{u}_3 = \mathcal{F}_3 & \text{in } \omega_T, \\
\textit{Initial conditions:} & \\
u_3(0) = u_{3,0},\ \dot{u}_3(0) = u_{3,1} & \text{in } \omega_T, \\
\textit{Boundary conditions:} & \\
\frac{2h^3}{3} \{ \rho \partial_\alpha \ddot{u}_3 \nu_\alpha - (\partial_\alpha \mathcal{M}^1_{\alpha\beta}) \nu_\beta - \partial_\tau (\mathcal{M}^1_{\alpha\beta} \nu_\alpha \tau_\beta) \} = \mathcal{G}_3 & \text{on } \gamma_1 \times (0,T), \\
-\mathcal{M}^1_{\alpha\beta} \nu_\alpha \nu_\beta = n_\alpha \nu_\alpha & \text{on } \gamma_1 \times (0,T), \\
u_3 = \partial_\nu u_3 = 0 & \text{on } \gamma_0 \times (0,T),
\end{array}
\right.
\qquad (FDP)^1
$$

where $\mathcal{F}_3 := \widetilde{f}_3 + \partial_\alpha m_\alpha$ and $\mathcal{G}_3 := \widetilde{g}_3 - m_\alpha \nu_\alpha + \partial_\tau (n_\alpha \tau_\alpha)$.

The thermo-electro-mechanical state $(\mathbf{u}_H(t), \phi(t), \vartheta(t))$ solves the following thermo-piezoelectric membrane differential problem:

$$
\left\{
\begin{array}{ll}
\textit{Field equations:} & \\
2h(\rho \ddot{u}_\alpha - \partial_\beta \mathcal{N}^1_{\alpha\beta}) = \widetilde{f}_\alpha + \widetilde{R}^1_{3\alpha\beta} [\![\partial_\beta \zeta]\!] & \text{in } \omega_T, \\
2h \partial_\alpha \widetilde{D}^1_\alpha = \widetilde{\rho}_e + \widetilde{\alpha}^1_{\alpha 3} [\![\partial_\alpha \zeta]\!] & \text{in } \omega_T, \\
2h(\partial_t \widetilde{\mathcal{S}}^1 + \partial_\alpha \widetilde{q}^1_\alpha) = \widetilde{r} + \widetilde{m}^1_3 [\![\dot{\zeta}]\!] & \text{in } \omega_T, \\
\textit{Initial conditions:} & \\
u_\alpha(0) = u_{\alpha,0},\ \dot{u}_\alpha(0) = u_{\alpha,1},\ \vartheta(0) = \vartheta_0 & \text{in } \omega_T, \\
\textit{Boundary conditions:} & \\
2h \mathcal{N}^1_{\alpha\beta} \nu_\beta = \widetilde{g}_\alpha - \widetilde{R}^1_{3\alpha\beta} \nu_\beta [\![\zeta]\!] & \text{on } \gamma_1 \times (0,T), \\
2h \widetilde{D}^1_\alpha \nu_\alpha = \widetilde{d} + \widetilde{\alpha}^1_{\alpha 3} \nu_\alpha [\![\zeta]\!] & \text{on } \gamma_1 \times (0,T), \\
2h \widetilde{q}^1_\alpha \nu_\alpha = \widetilde{\varrho} & \text{on } \gamma_1 \times (0,T), \\
u_\alpha = \phi = \vartheta = 0 & \text{on } \gamma_0 \times (0,T).
\end{array}
\right.
$$

It is important to remark that the information regarding the piezomagnetic behavior of the plate is now contained in the source terms appearing on the right-hand side of the equations, depending on the jump of the applied magnetic potentials at the upper and lower faces of the plate.

4.1.6.2 The Actuator-Sensor model

Theorem 4.5. *Under assumption (4.22), the sequence* $\{\mathcal{U}(\varepsilon)\}_{\varepsilon>0}$ *weakly converges to the limit* $\widetilde{\mathcal{U}} := (\tilde{\mathbf{u}}, \tilde{\varphi}, \tilde{\zeta}, \tilde{\theta})$ *in the space* $L^2(0,T;\mathbf{H}^1(\Omega)) \times L^2(0,T;\mathbb{X}(\Omega)) \times L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$.

Proof. Again, the proof relies on the uniform bound of the scaled energy $\{\mathcal{E}(\varepsilon)(t)\}_{\varepsilon>0}$, which can be established by procedures analogous to those used in steps (i), (ii) and (iii) of the proof of Theorem 4.3, taking into account the different scalings on the electric and magnetic potentials. This result implies that the sequences $\{\boldsymbol{\kappa}(\varepsilon)\}_{\varepsilon>0}$, $\{\boldsymbol{\chi}(\varepsilon)\}_{\varepsilon>0}$, $\{\bar{\boldsymbol{\tau}}(\varepsilon)\}_{\varepsilon>0}$ and $\{\boldsymbol{\gamma}(\varepsilon)\}_{\varepsilon>0}$ are uniformly bounded in $L^2(0,T;\mathbf{L}^2(\Omega))$, therefore we have the following weak convergences (up to a subsequence):

$$\begin{aligned}
&\kappa_{ij}(\varepsilon) \rightharpoonup \tilde{\kappa}_{ij} \quad \text{in } L^2(0,T;L^2(\Omega)),\\
&\bar{\tau}_i(\varepsilon) \rightharpoonup \tilde{\bar{\tau}}_i \quad \text{in } L^2(0,T;L^2(\Omega)),\\
&\chi_i(\varepsilon) \rightharpoonup \tilde{\chi}_i \quad \text{in } L^2(0,T;L^2(\Omega)),\\
&\gamma_i(\varepsilon) \rightharpoonup \tilde{\gamma}_i \quad \text{in } L^2(0,T;L^2(\Omega)).
\end{aligned}$$

Moreover, by means of Korn's and Poincaré's inequalities and from the definition of $\kappa_{ij}(\varepsilon)$, $\chi_i(\varepsilon)$ and $\gamma_i(\varepsilon)$, we infer that $\|\mathbf{u}(\varepsilon)\|_{1,\Omega}$, $\|\zeta(\varepsilon)\|_{1,\Omega}$ and $\|\theta(\varepsilon)\|_{1,\Omega}$ are also bounded, so that

$$\begin{aligned}
&\mathbf{u}(\varepsilon) \rightharpoonup \tilde{\mathbf{u}} \quad \text{in } L^2(0,T;\mathbf{H}^1(\Omega)),\\
&\dot{\mathbf{u}}(\varepsilon) \rightharpoonup \dot{\tilde{\mathbf{u}}} \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega)),\\
&\zeta(\varepsilon) \rightharpoonup \tilde{\zeta} \quad \text{in } L^2(0,T;H^1(\Omega)),\\
&\theta(\varepsilon) \rightharpoonup \tilde{\theta} \quad \text{in } L^2(0,T;H^1(\Omega)).
\end{aligned}$$

Upon writing

$$\bar{\varphi}(\varepsilon)(\tilde{x}, x_3) = \int_{-h}^{x_3} \partial_3\bar{\varphi}(\varepsilon)(\tilde{x}, y_3)dy_3,$$

it follows that $|\bar{\varphi}(\varepsilon)|_{0,\Omega} \leqslant 2h|\partial_3\bar{\varphi}(\varepsilon)|_{0,\Omega} \leqslant ce^{mT}$, by virtue of the boundedness of $\bar{\tau}_i(\varepsilon)$. This implies that both $\bar{\varphi}(\varepsilon)$ and $\varphi(\varepsilon)$ are bounded in $L^2(\Omega)$ and thus,

$$\begin{aligned}
&\bar{\varphi}(\varepsilon) \rightharpoonup \tilde{\bar{\varphi}} \quad \text{in } L^2(0,T;\mathbb{X}_0(\Omega)),\\
&\varphi(\varepsilon) \rightharpoonup \tilde{\varphi} \quad \text{in } L^2(0,T;\mathbb{X}(\Omega)).
\end{aligned}$$

This completes the proof. □

Theorem 4.6. *The weak limit* $\widetilde{\mathcal{U}}(t) = (\tilde{\mathbf{u}}(t), \tilde{\varphi}(t), \tilde{\zeta}(t), \tilde{\theta}(t))$ *is the solution to the limit variational problem:*

$$\left\{\begin{array}{l}
\textit{Find } \widetilde{\mathcal{U}}(t) \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}(\Omega) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0),\ t \in (0,T) \ \textit{ such that}\\
\widetilde{A}_2(\widetilde{\mathcal{U}}(t), \mathcal{V}) = \widetilde{L}_2(\mathcal{V}),\ \textit{for all } \mathcal{V} \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}_0(\Omega) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0),\\
\tilde{\varphi} = \varphi^{\pm} \textit{ on } \Gamma_{\pm},
\end{array}\right. \tag{4.30}$$

where

$$\widetilde{A}_2(\widetilde{\mathcal{U}}(t),\mathcal{V}) := \int_\Omega \Big\{\Big(\widetilde{C}^2_{\alpha\beta\sigma\tau}e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{P}^2_{3\alpha\beta}\partial_3\tilde{\varphi}(t) + \widetilde{R}^2_{\sigma\alpha\beta}\partial_\sigma\tilde{\zeta}(t) - \widetilde{\beta}^2_{\alpha\beta}\tilde{\theta}(t)\Big)\, e_{\alpha\beta}(\mathbf{v}) +$$
$$+ \Big(-\widetilde{P}^2_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}(t)) + \widetilde{\alpha}^2_{\alpha3}\partial_\alpha\tilde{\zeta}(t) + \widetilde{X}^2_{33}\partial_3\tilde{\varphi}(t) - \widetilde{p}^2_3\tilde{\theta}(t)\Big)\,\partial_3\psi +$$
$$+ \Big(-\widetilde{R}^2_{\alpha\sigma\tau}e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{M}^2_{\alpha\beta}\partial_\beta\tilde{\zeta}(t) + \widetilde{\alpha}^2_{\alpha3}\partial_3\tilde{\varphi}(t) - \widetilde{m}^2_\alpha\tilde{\theta}(t)\Big)\,\partial_\alpha\xi +$$
$$+ \Big(\widetilde{\beta}^2_{\alpha\beta}e_{\alpha\beta}(\dot{\tilde{\mathbf{u}}}(t)) - \widetilde{p}^2_3\partial_3\dot{\tilde{\varphi}}(t) - \widetilde{m}^2_\alpha\partial_\alpha\dot{\tilde{\zeta}}(t) + \widetilde{c}^2_v\dot{\tilde{\theta}}(t)\Big)\,\eta +$$
$$+ \widetilde{K}^1_{\alpha\beta}\partial_\beta\tilde{\theta}(t)\partial_\alpha\eta + \rho\ddot{\tilde{u}}_i(t)v_i\Big\}\,dx,$$
$$\widetilde{L}_2(\mathcal{V}) := (\mathbf{f},\mathbf{v}) + (\mathbf{g},\mathbf{v})_{\mathbf{L}^2(\widehat{\Gamma})} + (r,\eta) - \tfrac{1}{T_0}(\varrho,\eta)_{L^2(\widehat{\Gamma})} - (b,\xi)_{L^2(\widehat{\Gamma})}.$$

The reduced magneto-electro-thermo-elastic coefficients $\widetilde{C}^2_{\alpha\beta\sigma\tau}$, $\widetilde{M}^2_{\alpha\beta}$, $\widetilde{K}^1_{\alpha\beta}$, $\widetilde{R}^2_{\sigma\alpha\beta}$, $\widetilde{\beta}^2_{\alpha\beta}$, $\widetilde{m}^2_\alpha$ $\widetilde{p}^2_3$, $\widetilde{P}^2_{3\alpha\beta}$ *and* $\widetilde{\alpha}^2_{\alpha3}$ *are listed in Appendix 1.*

Proof. For the sake of clarity the proof is split into three parts.
(i) By the definition of $\kappa_{ij}(\varepsilon)$, $\bar{\tau}_i(\varepsilon)$, $\chi_i(\varepsilon)$ and $\gamma_i(\varepsilon)$, and thanks to the results of Theorem 4.5, there exists two constants C_M and C_K such that

$$\begin{array}{lll} |e_{\alpha\beta}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant C_M e^{C_K T}, & |e_{\alpha3}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}, & |e_{33}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant \varepsilon^2 C_M e^{C_K T}, \\ |\partial_\alpha\varphi(\varepsilon)|_{0,\Omega} \leqslant \frac{1}{\varepsilon}C_M e^{C_K T}, & |\partial_3\varphi(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T}, & \\ |\partial_\alpha\zeta(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T}, & |\partial_3\zeta(\varepsilon)|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}, & \\ |\partial_\alpha\theta(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T}, & |\partial_3\theta(\varepsilon)|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}. & \end{array} \quad (4.31)$$

From the first set of inequalities $(4.31)_1$, we get that $e_{i3}(\mathbf{u}(\varepsilon)(t)) \to 0$ in $L^2(\Omega)$ for almost every $t \in (0,T)$. Also, as $\mathbf{u}(\varepsilon)(t) \rightharpoonup \tilde{\mathbf{u}}(t)$ in $\mathbf{H}^1(\Omega)$, we have that $e_{i3}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup e_{i3}(\tilde{\mathbf{u}}(t))$ and so $e_{i3}(\tilde{\mathbf{u}}(t)) = 0$ by uniqueness of the limit. This implies that $\partial_3\tilde{u}_3 = 0$, i.e., $\tilde{u}_3(\tilde{x},x_3) = \tilde{u}_3(\tilde{x})$ is independent of x_3. We also have that $\partial_3\tilde{u}_\alpha = -\partial_\alpha\tilde{u}_3$, i.e., $\tilde{u}_\alpha(\tilde{x},x_3) = \tilde{u}_\alpha(\tilde{x}) - x_3\partial_\alpha\tilde{u}_3(\tilde{x})$. Consequently, $\tilde{\mathbf{u}}(t) \in \mathbf{V}_{KL}(\Omega)$. Moreover, we obtain that $e_{\alpha\beta}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{\alpha\beta}(t) = e_{\alpha\beta}(\tilde{\mathbf{u}}(t))$ in $L^2(\Omega)$, $\frac{1}{\varepsilon}e_{\alpha3}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{\alpha3}(t)$ in $L^2(\Omega)$, $\frac{1}{\varepsilon}e_{33}(\mathbf{u}(\varepsilon)(t)) \to 0$ in $L^2(\Omega)$ and, also, $\frac{1}{\varepsilon^2}e_{33}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{33}(t)$ in $L^2(\Omega)$.

From the second set of inequalities $(4.31)_2$, we have that $\varepsilon\partial_\alpha\varphi(\varepsilon)(t) \to 0$ in $L^2(\Omega)$ and $\partial_3\varphi(\varepsilon)(t) \rightharpoonup \partial_3\tilde{\varphi}(t)$ in $L^2(\Omega)$.

From the last sets of inequalities $(4.31)_{3,4}$, since $\zeta(\varepsilon)(t) \rightharpoonup \tilde{\zeta}(t)$ in $H^1(\Omega)$, we infer that $\partial_\alpha\zeta(\varepsilon)(t) \rightharpoonup \tilde{\chi}_\alpha(t) = \partial_\alpha\tilde{\zeta}(t)$ in $L^2(\Omega)$ and, also, $\partial_3\zeta(\varepsilon)(t) \to 0$ in $L^2(\Omega)$, i.e., $\tilde{\zeta}(t) = \tilde{\zeta}(\tilde{x})(t)$ is independent of x_3. Besides, $\frac{1}{\varepsilon}\partial_3\zeta(\varepsilon)(t) \rightharpoonup \chi_3(t)$ in $L^2(\Omega)$. Similarly, since $\theta(\varepsilon)(t) \rightharpoonup \tilde{\theta}(t)$ in $H^1(\Omega)$, we obtain that $\partial_\alpha\theta(\varepsilon)(t) \rightharpoonup \tilde{\gamma}_\alpha(t) = \partial_\alpha\tilde{\theta}(t)$ in $L^2(\Omega)$, $\partial_3\theta(\varepsilon)(t) \to 0$ in $L^2(\Omega)$, i.e., $\tilde{\theta}(t) = \tilde{\theta}(\tilde{x})(t)$ and, finally, $\frac{1}{\varepsilon}\partial_3\theta(\varepsilon)(t) \rightharpoonup \gamma_3(t)$ in $L^2(\Omega)$.
(ii) *Computations of* κ_{i3}, χ_3 *and* γ_3. Let us multiply problem (4.13) by ε^2 and let ε tend to zero. We get the following equation:

$$C_{3333}\kappa_{33} + 2C_{\alpha333}\kappa_{\alpha3} + R_{333}\chi_3 + C_{\alpha\beta33}e_{\alpha\beta}(\tilde{\mathbf{u}}) + R_{\alpha33}\partial_\alpha\tilde{\zeta} + P_{333}\partial_3\tilde{\varphi} - \beta_{33}\tilde{\theta} = 0.$$

By multiplying problem (4.13) by ε, choosing test functions $v_3 = \psi = \eta = \xi = 0$ and letting ε tend to zero, we have that

$$C_{\alpha333}\kappa_{33} + 2C_{\alpha3\beta3}\kappa_{\beta3} + R_{3\alpha3}\chi_3 + C_{\sigma\beta\alpha3}e_{\sigma\beta}(\tilde{\mathbf{u}}) + R_{\sigma\alpha3}\partial_\sigma\tilde{\zeta} + P_{3\alpha3}\partial_3\tilde{\varphi} - \beta_{\alpha3}\tilde{\theta} = 0.$$

Similarly, by multiplying problem (4.13) by ε and choosing test functions $v_i = \eta = \psi = 0$, when ε tends to zero, we find

$$-R_{333}\kappa_{33} - 2R_{3\alpha 3}\kappa_{\alpha 3} + M_{33}\chi_3 - R_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}) + M_{\alpha 3}\partial_\alpha\tilde{\zeta} + \alpha_{33}\partial_3\tilde{\varphi} - m_3\tilde{\theta} = 0.$$

Finally, if we multiply by ε and choose test functions $v_i = \psi = \xi = 0$, we obtain the last equation

$$K_{33}\gamma_3 + K_{\alpha 3}\partial_\alpha\tilde{\theta} = 0.$$

By combining the whole set of equations above we are now in a position to characterize κ_{i3}, χ_3 and γ_3. Let $\mathbf{l}^2 = (l_i^2)$ be the vector whose components are defined by $l_\alpha^2 := 2\kappa_{\alpha 3}$ and $l_3^2 := \kappa_{33}$, and $(d_{ij}) := (C_{i3j3})^{-1}$, then

$$\begin{aligned} l_i^2 &= -d_{ij}\left\{\left(C_{\alpha\beta j3} + \ell' R_{3j3}R'_{3\alpha\beta}\right)e_{\alpha\beta}(\tilde{\mathbf{u}}) + \left(R_{\alpha j3} - \ell' R_{3j3}M'_{\alpha 3}\right)\partial_\alpha\tilde{\zeta} +\right.\\ &\quad\left. + \left(P_{3j3} - \ell' R_{3j3}\alpha'_{33}\right)\partial_3\tilde{\varphi} - \left(\beta_{j3} - \ell' R_{3j3}m'_3\right)\tilde{\theta}\right\},\\ \chi_3 &= \ell'\left(R'_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}) - M'_{\alpha 3}\partial_\alpha\tilde{\zeta} - \alpha'_{33}\partial_3\tilde{\varphi} + m'_3\tilde{\theta}\right),\\ \gamma_3 &= -K'_{\alpha 3}\partial_\alpha\tilde{\theta}. \end{aligned} \tag{4.32}$$

Coefficients ℓ', $R'_{3\alpha\beta}$, $M'_{\alpha 3}$, α'_{33}, m'_3 and $K'_{\alpha 3}$ are defined in Appendix 1.

(iii) *Definition of the limit problem.* We let test functions be $\mathcal{V} = (\mathbf{v}, \psi, \xi, \eta) \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}_0(\Omega) \times H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0)$ in problem (4.13) and let $\varepsilon \to 0$, by substituting expressions (4.32), we obtain, as customary, the limit evolution problem

$$\begin{cases} \text{Find } \widetilde{\mathcal{U}}(t) \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}(\Omega) \times H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0),\ t \in (0, T) \text{ such that}\\ \displaystyle\int_\Omega \left\{\left(\widetilde{C}^2_{\alpha\beta\sigma\tau}e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{P}^2_{3\alpha\beta}\partial_3\tilde{\varphi}(t) + \widetilde{R}^2_{\sigma\alpha\beta}\partial_\sigma\tilde{\zeta}(t) - \widetilde{\beta}^2_{\alpha\beta}\tilde{\theta}(t)\right)e_{\alpha\beta}(\mathbf{v}) +\right.\\ \quad + \left(-\widetilde{P}^2_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}(t)) + \widetilde{\alpha}^2_{\alpha 3}\partial_\alpha\tilde{\zeta}(t) + \widetilde{X}^2_{33}\partial_3\tilde{\varphi}(t) - \widetilde{p}^2_3\tilde{\theta}(t)\right)\partial_3\psi +\\ \quad + \left(-\widetilde{R}^2_{\alpha\sigma\tau}e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{M}^2_{\alpha\beta}\partial_\beta\tilde{\zeta}(t) + \widetilde{\alpha}^2_{\alpha 3}\partial_3\tilde{\varphi}(t) - \widetilde{m}^2_\alpha\tilde{\theta}(t)\right)\partial_\alpha\xi +\\ \quad + \left(\widetilde{\beta}^2_{\alpha\beta}e_{\alpha\beta}(\dot{\tilde{\mathbf{u}}}(t)) - \widetilde{p}^2_3\partial_3\dot{\tilde{\varphi}}(t) - \widetilde{m}^2_\alpha\partial_\alpha\dot{\tilde{\zeta}}(t) + \widetilde{c}^2_v\dot{\tilde{\theta}}(t)\right)\eta +\\ \quad \left. + \widetilde{K}^1_{\alpha\beta}\partial_\beta\tilde{\theta}(t)\partial_\alpha\eta + \rho\ddot{\tilde{u}}_i(t)v_i\right\}dx = \widetilde{L}_2(\mathcal{V}),\\ \tilde{\varphi} = \varphi^\pm \text{ on } \Gamma_\pm,\ \text{for all } \mathcal{V} \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}_0(\Omega) \times H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0) \end{cases} \tag{4.33}$$

Problem (4.33) is formally equivalent to (4.30), presented in the statement of Theorem 4.6.

□

The limit evolution problem

In this section we present the variational and differential formulations of the evolution problem for an actuator-sensor magneto-electro-thermo-elastic plate. The primary unknowns of the limit problem are collected into the limit magneto-electro-thermo-elastic state $\widetilde{\mathcal{U}}(t) = (\tilde{\mathbf{u}}(t), \tilde{\varphi}(t), \tilde{\zeta}(t), \tilde{\theta}(t)) \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}(\Omega) \times H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0)$, whose components take the following form:

$$\begin{aligned}
&\tilde{u}_\alpha(\tilde{x}, x_3) = u_\alpha(\tilde{x}) - x_3\partial_\alpha u_3(\tilde{x}), \ \ \mathbf{u}_H := (u_\alpha),\\
&\tilde{u}_3(\tilde{x}, x_3) = u_3(\tilde{x}),\\
&\tilde{\zeta}(\tilde{x}, x_3) = \varsigma(\tilde{x}),\\
&\tilde{\theta}(\tilde{x}, x_3) = \vartheta(\tilde{x}).
\end{aligned} \tag{4.34}$$

In the case of a homogeneous material, the reduced magneto-electro-thermo-elastic coefficients $\widetilde{C}^2_{\alpha\beta\sigma\tau}$, $\widetilde{M}^2_{\alpha\beta}$, $\widetilde{K}^1_{\alpha\beta}$, $\widetilde{R}^2_{\sigma\alpha\beta}$, $\widetilde{\beta}^2_{\alpha\beta}$, $\widetilde{m}^2_\alpha$ $\widetilde{p}^2_3$, $\widetilde{P}^2_{3\alpha\beta}$, $\widetilde{X}^2_{33}$ and $\widetilde{\alpha}^2_{\alpha 3}$ are constant functions and, thus, the limit evolution problem decouples into two variational subproblems, namely, the flexural problem, which gives u_3, and the thermo-piezomagnetic membrane problem, which gives the triplet $(\mathbf{u}_H, \varsigma, \vartheta)$. Moreover, we can characterize explicitly the limit electric potential $\tilde{\varphi}$ as a second order polynomial function of x_3.

Indeed, by choosing test functions $\mathcal{V} = (\mathbf{0}, \psi, 0, 0)$ in (4.33), after an integration by parts along x_3, we get the expression of the limit electric potential $\tilde{\varphi}$:

$$\tilde{\varphi}(\tilde{x}, x_3) = \sum_{k=0}^{2} f^k(\tilde{x}) x_3^k, \tag{4.35}$$

where

$$f^0 = \langle\varphi\rangle + \frac{h^2 \widetilde{P}^2_{3\alpha\beta}}{2\widetilde{X}^2_{33}}\partial_{\alpha\beta}u_3, \ \ f^1 = \frac{[\![\varphi]\!]}{2h}, \ \ f^2 = -\frac{\widetilde{P}^2_{3\alpha\beta}}{2\widetilde{X}^2_{33}}\partial_{\alpha\beta}u_3.$$

with $\langle\varphi\rangle := \frac{\varphi^+ + \varphi^-}{2}$ and $[\![\varphi]\!] := \varphi^+ - \varphi^-$ representing, respectively, the mean value and the jump function between the top and bottom boundary values of φ. As the careful reader can notice, the limit electric potential becomes a known function depending on the transversal displacement u_3 of the plate and on the values of the magnetic potentials $\varphi^\pm$, applied on the upper and lower surfaces $\Gamma_\pm$.

By virtue of the characterization (4.35) of the limit electric potential $\tilde{\varphi}$, we can now rewrite the limit evolution problem. After an integration by parts along x_3, we obtain a two-dimensional problem defined over the middle surface ω of the plate:

$$\left\{\begin{aligned}
&\text{Find } \widetilde{\mathcal{U}} = (\mathbf{u}_H(t), u_3(t), \varsigma(t), \vartheta(t)) \in \mathbf{V}_H(\omega, \gamma_0)\times V_3(\omega, \gamma_0)\times\\
&\times H^1(\omega, \gamma_0)\times H^1(\omega, \gamma_0) \text{ such that}\\
&2h\int_\omega \left\{\mathcal{N}^2_{\alpha\beta}(\mathbf{u}_H(t), \varsigma(t), \vartheta(t)) e_{\alpha\beta}(\mathbf{v}_H) + \rho\ddot{u}_\alpha(t)v_\alpha\right\} d\tilde{x}+\\
&+2h\int_\omega \left\{-\widetilde{\mathcal{B}}^2_\alpha(\mathbf{u}_H(t), \varsigma(t), \vartheta(t))\partial_\alpha\xi + \partial_t\widetilde{\mathcal{S}}^2(\mathbf{u}_H(t), \varsigma(t), \vartheta(t))\eta - \widetilde{q}^1_\alpha(\vartheta(t))\partial_\alpha\eta\right\} d\tilde{x}+\\
&+\frac{2h^3}{3}\int_\omega \left\{\mathcal{M}^2_{\alpha\beta}(u_3(t))\partial_{\alpha\beta}v_3 + \rho\partial_\alpha\ddot{w}(t)\partial_\alpha v_3 + \frac{3}{h^2}\rho\ddot{w}(t)v_3\right\} d\tilde{x} = \mathcal{L}_2(\widetilde{\mathcal{V}}),\\
&\text{for all } \widetilde{\mathcal{V}} = (\mathbf{v}_H, v_3, \xi, \eta) \in \mathbf{V}_H(\omega, \gamma_0) \times V_3(\omega, \gamma_0) \times H^1(\omega, \gamma_0) \times H^1(\omega, \gamma_0),
\end{aligned}\right.$$

with

$$\begin{aligned}
\mathcal{L}_2(\widetilde{\mathcal{V}}) := &\int_\omega \left\{\widetilde{f}_i v_i - m_\alpha\partial_\alpha v_3 + \widetilde{r}\eta\right\} d\tilde{x} + \int_{\gamma_1} \left\{\widetilde{g}_i v_i - n_\alpha\partial_\alpha v_3 - \widetilde{b}\xi - \widetilde{\varrho}\eta\right\} d\gamma-\\
&-\int_\omega \left\{\widetilde{P}^2_{3\alpha\beta}[\![\varphi]\!] e_{\alpha\beta}(\mathbf{v}_H) + \widetilde{\alpha}^2_{\alpha 3}[\![\varphi]\!]\partial_\alpha\xi - \widetilde{p}^2_3[\![\dot{\varphi}]\!]\eta\right\} d\tilde{x}.
\end{aligned}$$

$(\mathcal{N}^2_{\alpha\beta})$, $(\mathcal{M}^2_{\alpha\beta})$, $(\widetilde{B}^2_\alpha)$, $\widetilde{\mathcal{S}}^2$ and $(\widetilde{q}^1_\alpha)$ represent, respectively, the membrane stress tensor, the moment tensor, the reduced magnetic induction, the reduced thermodynamic entropy and the reduced heat flow vector of the plate, whose components are defined by the constitutive laws below:

$$\left\{\begin{array}{l} \mathcal{N}^2_{\alpha\beta} := \widetilde{C}^2_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}_H) + \widetilde{R}^2_{\sigma\alpha\beta}\partial_\sigma\varsigma - \widetilde{\beta}^2_{\alpha\beta}\vartheta, \\ \widetilde{B}^2_\alpha := \widetilde{R}^2_{\alpha\sigma\tau} e_{\sigma\tau}(\mathbf{u}_H) - \widetilde{M}^2_{\alpha\beta}\partial_\beta\varsigma + \widetilde{m}^2_\alpha\vartheta, \\ \widetilde{\mathcal{S}}^2 := \widetilde{\beta}^2_{\alpha\beta} e_{\alpha\beta}(\mathbf{u}_H) - \widetilde{m}^2_\alpha\partial_\alpha\varsigma + \widetilde{c}^2_v\vartheta, \\ \mathcal{M}^2_{\alpha\beta} := \widetilde{A}^2_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}u_3, \\ \widetilde{q}^1_\alpha := -\frac{1}{T_0}\widetilde{K}^1_{\alpha\beta}\partial_\beta\vartheta, \end{array}\right. \tag{4.36}$$

where $\widetilde{A}^2_{\alpha\beta\sigma\tau} := \widetilde{C}^2_{\alpha\beta\sigma\tau} + \frac{\widetilde{P}^2_{3\alpha\beta}\widetilde{P}^2_{3\sigma\tau}}{\widetilde{X}^2_{33}}$. The variational problem above can be split into two two-dimensional decoupled problems: namely, the flexural problem and the thermo-piezomagnetic membrane problem.

The flexural problem reads as follows:

$$\left\{\begin{array}{l} \text{Find } u_3(t) \in V_3(\omega,\gamma_0),\ t \in (0,T) \text{ such that} \\ \dfrac{2h^3}{3}\displaystyle\int_\omega \left\{\mathcal{M}^2_{\alpha\beta}(u_3(t))\partial_{\alpha\beta}v_3 + \rho\partial_\alpha\ddot{u}_3(t)\partial_\alpha v_3 + \frac{3}{h^2}\rho\ddot{u}_3(t)v_3\right\} d\tilde{x} = \\ \quad = \displaystyle\int_\omega \left\{\widetilde{f}_3 v_3 - m_\alpha\partial_\alpha v_3\right\} d\tilde{x} + \int_{\gamma_1} \{\widetilde{g}_3 v_3 - n_\alpha\partial_\alpha v_3\}\, d\gamma, \\ \text{for all } v_3 \in V_3(\omega,\gamma_0). \end{array}\right.$$

The two-dimensional thermo-piezomagnetic membrane problem takes the following form

$$\left\{\begin{array}{l} \text{Find } (\mathbf{u}_H(t),\varsigma(t),\vartheta(t)), \in \mathbf{V}_H(\omega,\gamma_0)\times H^1(\omega,\gamma_0)\times H^1(\omega,\gamma_0),\ t\in(0,T) \\ \text{such that } 2h\displaystyle\int_\omega \left\{\mathcal{N}^2_{\alpha\beta}(\mathbf{u}_H(t),\varsigma(t),\vartheta(t))e_{\alpha\beta}(\mathbf{v}_H) + \rho\ddot{u}_\alpha(t)v_\alpha\right\} d\tilde{x} + \\ +2h\displaystyle\int_\omega \left\{-\widetilde{B}^2_\alpha(\mathbf{u}_H(t),\varsigma(t),\vartheta(t))\partial_\alpha\xi + \partial_t\widetilde{\mathcal{S}}^2(\mathbf{u}_H(t),\varsigma(t),\vartheta(t))\eta - \widetilde{q}^1_\alpha(\vartheta(t))\partial_\alpha\eta\right\} d\tilde{x} \\ = \displaystyle\int_\omega \left\{\widetilde{f}_\alpha v_\alpha - \widetilde{P}^2_{3\alpha\beta}[\![\varphi]\!]e_{\alpha\beta}(\mathbf{v}_H) - \widetilde{\alpha}^2_{\alpha 3}[\![\varphi]\!]\partial_\alpha\xi + (\widetilde{r} + \widetilde{p}^2_3[\![\dot{\varphi}]\!])\eta\right\} d\tilde{x} + \\ + \displaystyle\int_{\gamma_1} \left\{\widetilde{g}_\alpha v_\alpha - \widetilde{b}\xi - \widetilde{\varrho}\eta\right\} d\gamma, \\ \text{for all } (\mathbf{v}_H,\xi,\eta) \in \mathbf{V}_H(\omega,\gamma_0)\times H^1(\omega,\gamma_0)\times H^1(\omega,\gamma_0). \end{array}\right.$$

We are now in a position to write the decoupled flexural and thermo-piezomagnetic membrane problems into their differential form by using Green's formulae on ω.

The transversal displacement u_3 solves a flexural differential problem $(FDP)^2$ analogous to $(FDP)^1$ with $\mathcal{M}^1_{\alpha\beta}$ replaced by $\mathcal{M}^2_{\alpha\beta}$ defined by $(4.36)_4$.

The thermo-magneto-elastic state $(\mathbf{u}_H(t),\varsigma(t),\vartheta(t))$ solves the following thermo-piezomagnetic membrane differential problem:

$$\begin{cases} \textit{Field equations:} \\ 2h(\rho \ddot{u}_\alpha - \partial_\beta \mathcal{N}^2_{\alpha\beta}) = \widetilde{f}_\alpha + \widetilde{P}^2_{3\alpha\beta}[\![\partial_\beta \varphi]\!] & \text{in } \omega_T, \\ 2h\partial_\alpha \widetilde{B}^2_\alpha = \widetilde{\alpha}^2_{\alpha 3}[\![\partial_\alpha \varphi]\!] & \text{in } \omega_T, \\ 2h(\partial_t \widetilde{S}^2 + \partial_\alpha \widetilde{q}^1_\alpha) = \widetilde{r} + \widetilde{p}^2_3[\![\dot{\varphi}]\!] & \text{in } \omega_T, \\ \textit{Initial conditions:} \\ u_\alpha(0) = u_{\alpha,0},\ \dot{u}_\alpha(0) = u_{\alpha,1},\ \vartheta(0) = \vartheta_0 & \text{in } \omega_T, \\ \textit{Boundary conditions:} \\ 2h\mathcal{N}^2_{\alpha\beta}\nu_\beta = \widetilde{g}_\alpha - \widetilde{P}^2_{3\alpha\beta}\nu_\beta[\![\varphi]\!] & \text{on } \gamma_1 \times (0,T), \\ 2h\widetilde{B}^2_\alpha \nu_\alpha = \widetilde{b} + \widetilde{\alpha}^2_{\alpha 3}\nu_\alpha[\![\varphi]\!] & \text{on } \gamma_1 \times (0,T), \\ 2h\widetilde{q}^1_\alpha \nu_\alpha = \widetilde{\varrho} & \text{on } \gamma_1 \times (0,T), \\ u_\alpha = \varsigma = \vartheta = 0 & \text{on } \gamma_0 \times (0,T). \end{cases} \tag{4.37}$$

It is important to remark that the information regarding the piezoelectric behavior of the plate is now contained in the source terms appearing on the right-hand side of the equations, depending on the jump of the applied electric potentials at the upper and lower faces of the plate.

4.1.6.3 The Actuator Model

Theorem 4.7. *Under assumption (4.22), the sequence* $\{\mathcal{U}(\varepsilon)\}_{\varepsilon>0}$ *weakly converges to the limit* $\widetilde{\mathcal{U}} := (\tilde{\mathbf{u}}, \tilde{\varphi}, \tilde{\zeta}, \tilde{\theta})$ *in the space* $L^2(0,T;\mathbf{H}^1(\Omega)) \times L^2(0,T;\mathbb{X}(\Omega)) \times L^2(0,T;\mathbb{X}(\Omega)) \times L^2(0,T;H^1(\Omega))$.

Proof. As usual, one can determine a uniform bound on the scaled energy $\{\mathcal{E}(\varepsilon)(t)\}_{\varepsilon>0}$ by using the same techniques as in steps (i), (ii) and (iii) of the proof of Theorem 4.3, taking into account the different scalings on the electric and magnetic potentials. From the bound on the energy we infer that the sequences $\{\boldsymbol{\kappa}(\varepsilon)\}_{\varepsilon>0}$, $\{\bar{\boldsymbol{\chi}}(\varepsilon)\}_{\varepsilon>0}$, $\{\bar{\boldsymbol{\tau}}(\varepsilon)\}_{\varepsilon>0}$ and $\{\boldsymbol{\gamma}(\varepsilon)\}_{\varepsilon>0}$ are bounded independently of ε in $L^2(0,T;\mathbf{L}^2(\Omega))$, meaning that $\kappa_{ij}(\varepsilon) \rightharpoonup \tilde{\kappa}_{ij}$, $\bar{\chi}_i(\varepsilon) \rightharpoonup \tilde{\bar{\chi}}_i$, $\bar{\tau}_i(\varepsilon) \rightharpoonup \tilde{\bar{\tau}}_i$ and $\gamma_i(\varepsilon) \rightharpoonup \tilde{\gamma}_i$ in $L^2(0,T;L^2(\Omega))$.

Besides, by means of Korn's and Poincaré's inequalities and from the definition of $\kappa_{ij}(\varepsilon)$ and $\gamma_i(\varepsilon)$, we infer that $\|\mathbf{u}(\varepsilon)\|_{1,\Omega}$ and $\|\theta(\varepsilon)\|_{1,\Omega}$ are also bounded, so that

$$\begin{aligned} \mathbf{u}(\varepsilon) &\rightharpoonup \tilde{\mathbf{u}} \quad \text{in } L^2(0,T;\mathbf{H}^1(\Omega)), \\ \dot{\mathbf{u}}(\varepsilon) &\rightharpoonup \dot{\tilde{\mathbf{u}}} \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega)), \\ \theta(\varepsilon) &\rightharpoonup \tilde{\theta} \quad \text{in } L^2(0,T;H^1(\Omega)). \end{aligned}$$

Since

$$\bar{\zeta}(\varepsilon)(\tilde{x}, x_3) = \int_{-h}^{x_3} \partial_3 \bar{\zeta}(\varepsilon)(\tilde{x}, y_3) dy_3 \ \text{ and } \ \bar{\varphi}(\varepsilon)(\tilde{x}, x_3) = \int_{-h}^{x_3} \partial_3 \bar{\varphi}(\varepsilon)(\tilde{x}, y_3) dy_3$$

we obtain that $|\bar{\zeta}(\varepsilon)|_{0,\Omega} \leqslant 2h|\partial_3 \bar{\zeta}(\varepsilon)|_{0,\Omega} \leqslant ce^{mT}$ and $|\bar{\varphi}(\varepsilon)|_{0,\Omega} \leqslant 2h|\partial_3 \bar{\varphi}(\varepsilon)|_{0,\Omega} \leqslant ce^{mT}$, by means of the boundedness of $\bar{\chi}_i(\varepsilon)$ and $\bar{\tau}_i(\varepsilon)$. This implies that $\bar{\zeta}(\varepsilon)$, $\zeta(\varepsilon)$,

$\bar{\varphi}(\varepsilon)$ and $\varphi(\varepsilon)$ are all bounded in $L^2(\Omega)$ and, thus,

$$\begin{aligned} \bar{\zeta}(\varepsilon) &\rightharpoonup \tilde{\bar{\zeta}} \quad \text{in } L^2(0,T;\mathbb{X}_0(\Omega)),\\ \zeta(\varepsilon) &\rightharpoonup \tilde{\zeta} \quad \text{in } L^2(0,T;\mathbb{X}(\Omega)),\\ \bar{\varphi}(\varepsilon) &\rightharpoonup \tilde{\bar{\varphi}} \quad \text{in } L^2(0,T;\mathbb{X}_0(\Omega)),\\ \varphi(\varepsilon) &\rightharpoonup \tilde{\varphi} \quad \text{in } L^2(0,T;\mathbb{X}(\Omega)). \end{aligned}$$

This completes the proof. □

Theorem 4.8. *The weak limit $\widetilde{\mathcal{U}}(t) = (\tilde{\mathbf{u}}(t), \tilde{\varphi}(t), \tilde{\zeta}(t), \tilde{\theta}(t))$ is the solution to the limit variational problem:*

$$\begin{cases} \text{Find } \widetilde{\mathcal{U}}(t) \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}(\Omega) \times \mathbb{X}(\Omega) \times H^1(\Omega,\Gamma_0),\ t \in (0,T) \text{ such that}\\ \widetilde{A}_3(\widetilde{\mathcal{U}}(t),\mathcal{V}) = \widetilde{L}_3(\mathcal{V}), \text{ for all } \mathcal{V} \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}_0(\Omega) \times \mathbb{X}_0(\Omega) \times H^1(\Omega,\Gamma_0),\\ \tilde{\varphi} = \varphi^{\pm},\ \tilde{\zeta} = \zeta^{\pm} \text{ on } \Gamma_{\pm}, \end{cases} \tag{4.38}$$

where

$$\begin{aligned} \widetilde{A}_3(\widetilde{\mathcal{U}}(t),\mathcal{V}) := \int_\Omega \Big\{ &\Big(\widetilde{C}^3_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{P}^3_{3\alpha\beta}\partial_3\tilde{\varphi}(t) + \widetilde{R}^3_{3\alpha\beta}\partial_3\tilde{\zeta}(t) - \widetilde{\beta}^3_{\alpha\beta}\tilde{\theta}(t)\Big) e_{\alpha\beta}(\mathbf{v}) +\\ &+ \Big(-\widetilde{P}^3_{3\alpha\beta} e_{\alpha\beta}(\tilde{\mathbf{u}}(t)) + \widetilde{X}^3_{33}\partial_3\tilde{\varphi}(t) + \widetilde{\alpha}^3_{33}\partial_3\tilde{\zeta}(t) - \widetilde{p}^3_3\tilde{\theta}(t)\Big)\partial_3\psi +\\ &+ \Big(-\widetilde{R}^3_{3\alpha\beta} e_{\alpha\beta}(\tilde{\mathbf{u}}(t)) + \widetilde{\alpha}^3_{33}\partial_3\tilde{\varphi}(t) + \widetilde{M}^3_{33}\partial_3\tilde{\zeta}(t) - \widetilde{m}^3_3\tilde{\theta}(t)\Big)\partial_3\xi +\\ &+ \Big(\widetilde{\beta}^3_{\alpha\beta} e_{\alpha\beta}(\dot{\tilde{\mathbf{u}}}(t)) - \widetilde{m}^3_3\partial_3\dot{\tilde{\zeta}}(t) - \widetilde{p}^3_3\partial_3\dot{\tilde{\varphi}}(t) + \widetilde{c}^3_v\dot{\tilde{\theta}}(t)\Big)\eta +\\ &+ \widetilde{K}^1_{\alpha\beta}\partial_\beta\tilde{\theta}(t)\partial_\alpha\eta + \rho\ddot{\tilde{u}}_i(t)v_i\Big\}\,dx,\\ \widetilde{L}_3(\mathcal{V}) := (\mathbf{f},\mathbf{v}) + (\mathbf{g},\mathbf{v})&_{\mathbf{L}^2(\hat{\Gamma})} + (r,\eta) - \tfrac{1}{T_0}(\varrho,\eta)_{L^2(\hat{\Gamma})} + (\rho_e,\psi). \end{aligned}$$

The reduced magneto-electro-thermo-elastic coefficients $\widetilde{C}^3_{\alpha\beta\sigma\tau}$, $\widetilde{X}^3_{33}$, $\widetilde{M}^3_{33}$, $\widetilde{P}^3_{3\alpha\beta}$, $\widetilde{\beta}^3_{\alpha\beta}$, $\widetilde{p}^3_3$, $\widetilde{m}^3_3$, $\widetilde{R}^3_{3\alpha\beta}$, $\widetilde{\alpha}^3_{33}$ and $\widetilde{c}^3_v$ are listed in Appendix 1.

Proof. For the sake of simplicity the proof is split into three parts, numbered from (i) to (iii).
(i) From definition of $\kappa_{ij}(\varepsilon)$, $\bar{\chi}_i(\varepsilon)$, $\bar{\tau}_i(\varepsilon)$ and $\gamma_i(\varepsilon)$, and thanks to the results of Theorem 4.7, there exist two constants C_M and C_K such that

$$\begin{aligned} &|e_{\alpha\beta}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant C_M e^{C_K T}, && |e_{\alpha 3}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}, && |e_{33}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant \varepsilon^2 C_M e^{C_K T},\\ &|\partial_\alpha\zeta(\varepsilon)|_{0,\Omega} \leqslant \tfrac{1}{\varepsilon} C_M e^{C_K T}, && |\partial_3\zeta(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T},\\ &|\partial_\alpha\varphi(\varepsilon)|_{0,\Omega} \leqslant \tfrac{1}{\varepsilon} C_M e^{C_K T}, && |\partial_3\varphi(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T},\\ &|\partial_\alpha\theta(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T}, && |\partial_3\theta(\varepsilon)|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}. \end{aligned} \tag{4.39}$$

From the first set of inequalities $(4.39)_1$, we get that $e_{i3}(\mathbf{u}(\varepsilon)(t)) \to 0$ in $L^2(\Omega)$ for almost every $t \in (0,T)$, and, since $\mathbf{u}(\varepsilon)(t) \rightharpoonup \tilde{\mathbf{u}}(t)$ in $\mathbf{H}^1(\Omega)$, we have that $e_{i3}(\tilde{\mathbf{u}}(t)) = 0$. Consequently, $\tilde{\mathbf{u}}(t) \in \mathbf{V}_{KL}(\Omega)$. Moreover, we obtain that $e_{\alpha\beta}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{\alpha\beta}(t) = e_{\alpha\beta}(\tilde{\mathbf{u}}(t))$ in $L^2(\Omega)$, $\frac{1}{\varepsilon}e_{\alpha 3}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{\alpha 3}(t)$ in $L^2(\Omega)$, $\frac{1}{\varepsilon}e_{33}(\mathbf{u}(\varepsilon)(t)) \to 0$ in $L^2(\Omega)$ and, also, $\frac{1}{\varepsilon^2}e_{33}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{33}(t)$ in $L^2(\Omega)$.

From the second and third sets of inequalities $(4.39)_{2,3}$, we have that $\varepsilon\partial_\alpha\zeta(\varepsilon)(t) \to 0$ in $L^2(\Omega)$, $\partial_3\zeta(\varepsilon)(t) \rightharpoonup \partial_3\tilde{\zeta}$ in $L^2(\Omega)$, $\varepsilon\partial_\alpha\varphi(\varepsilon)(t) \to 0$ in $L^2(\Omega)$, $\partial_3\varphi(\varepsilon)(t) \rightharpoonup \partial_3\tilde{\varphi}(t)$ in $L^2(\Omega)$.

From the last set of inequalities $(4.39)_4$, since $\theta(\varepsilon)(t) \rightharpoonup \tilde{\theta}(t)$ in $H^1(\Omega)$, we obtain that $\partial_\alpha\theta(\varepsilon)(t) \rightharpoonup \tilde{\gamma}_\alpha(t) = \partial_\alpha\tilde{\theta}(t)$ in $L^2(\Omega)$, $\partial_3\theta(\varepsilon)(t) \to 0$ in $L^2(\Omega)$, i.e., $\tilde{\theta}(t) = \tilde{\theta}(\tilde{x})(t)$ and, finally, $\frac{1}{\varepsilon}\partial_3\theta(\varepsilon)(t) \rightharpoonup \gamma_3(t)$ in $L^2(\Omega)$.
(ii) *Computations of* κ_{i3} *and* γ_3. Let us multiply problem (4.19) by ε^2 and let ε tend to zero. We get the following equation:

$$C_{3333}\kappa_{33} + 2C_{\alpha333}\kappa_{\alpha3} + C_{\alpha\beta33}e_{\alpha\beta}(\tilde{\mathbf{u}}) + P_{333}\partial_3\tilde{\varphi} + R_{333}\partial_3\tilde{\zeta} - \beta_{33}\tilde{\theta} = 0.$$

By multiplying problem (4.19) by ε, we choose test functions $v_3 = \psi = \eta = \xi = 0$ and we let ε tend to zero, we have that

$$C_{\alpha333}\kappa_{33} + 2C_{\alpha3\beta3}\kappa_{\beta3} + C_{\sigma\beta\alpha3}e_{\sigma\beta}(\tilde{\mathbf{u}}) + P_{3\alpha3}\partial_3\tilde{\varphi} + R_{3\alpha3}\partial_3\tilde{\zeta} - \beta_{\alpha3}\tilde{\theta} = 0.$$

Finally, if we multiply by ε and choose test functions $v_i = \psi = \xi = 0$, we obtain the last equation

$$K_{33}\gamma_3 + K_{\alpha3}\partial_\alpha\tilde{\theta} = 0.$$

By solving the linear system above, we can characterize κ_{i3} and γ_3. Let $\mathbf{l}^3 = (l_i^3)$ be the vector whose components are defined by $l_\alpha^3 := 2\kappa_{\alpha3}$ and $l_3^3 := \kappa_{33}$, and $(d_{ij}) := (C_{i3j3})^{-1}$, then

$$\begin{aligned} l_i^3 &= -d_{ij}\left(C_{\alpha\beta j3}e_{\alpha\beta}(\tilde{\mathbf{u}}) + P_{3j3}\partial_3\tilde{\varphi} + R_{3j3}\partial_3\tilde{\zeta} - \beta_{j3}\tilde{\theta}\right), \\ \gamma_3 &= -K'_{\alpha3}\partial_\alpha\tilde{\theta}. \end{aligned} \tag{4.40}$$

(iii) *Definition of the limit problem.* By choosing test functions $\mathcal{V} = (\mathbf{v}, \psi, \xi, \eta) \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}_0(\Omega) \times \mathbb{X}_0(\Omega) \times H^1(\omega, \gamma_0)$ in problem (4.19) and let $\varepsilon \to 0$, by substituting expressions (4.40), we obtain, as customary, the limit evolution problem

$$\left\{\begin{aligned} &\text{Find } \tilde{\mathcal{U}}(t) \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}(\Omega) \times \mathbb{X}(\Omega) \times H^1(\Omega, \Gamma_0),\ t \in (0,T) \text{ such that} \\ &\int_\Omega \Big\{ \Big(\tilde{C}^3_{\alpha\beta\sigma\tau}e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \tilde{P}^3_{3\alpha\beta}\partial_3\tilde{\varphi}(t) + \tilde{R}^3_{3\alpha\beta}\partial_3\tilde{\zeta}(t) - \tilde{\beta}^3_{\alpha\beta}\tilde{\theta}(t)\Big) e_{\alpha\beta}(\mathbf{v}) + \\ &\quad + \Big(-\tilde{P}^3_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}(t)) + \tilde{X}^3_{33}\partial_3\tilde{\varphi}(t) + \tilde{\alpha}^3_{33}\partial_3\tilde{\zeta}(t) - \tilde{p}^3_3\tilde{\theta}(t)\Big)\partial_3\psi + \\ &\quad + \Big(-\tilde{R}^3_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}(t)) + \tilde{\alpha}^3_{33}\partial_3\tilde{\varphi}(t) + \tilde{M}^3_{33}\partial_3\tilde{\zeta}(t) - \tilde{m}^3_3\tilde{\theta}(t)\Big)\partial_3\xi + \\ &\quad + \Big(\tilde{\beta}^3_{\alpha\beta}e_{\alpha\beta}(\dot{\tilde{\mathbf{u}}}(t)) - \tilde{m}^3_3\partial_3\dot{\tilde{\zeta}}(t) - \tilde{p}^3_3\partial_3\dot{\tilde{\varphi}}(t) + \tilde{c}^3_v\dot{\tilde{\theta}}(t)\Big)\eta + \\ &\quad + \tilde{K}^1_{\alpha\beta}\partial_\beta\tilde{\theta}(t)\partial_\alpha\eta + \rho\ddot{\tilde{u}}_i(t)v_i\Big\}\, dx = \tilde{L}_3(\mathcal{V}), \\ &\tilde{\varphi} = \varphi^\pm,\ \tilde{\zeta} = \zeta^\pm \text{ on } \Gamma_\pm, \text{ for all } \mathcal{V} \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}_0(\Omega) \times \mathbb{X}_0(\Omega) \times H^1(\Omega, \Gamma_0). \end{aligned}\right. \tag{4.41}$$

Thus, the result is achieved.

□

The limit evolution problem

In this section we present the variational and differential formulations of the evolution problem for an actuator magneto-electro-thermo-elastic plate. The primary unknowns $\tilde{\mathcal{U}} = (\tilde{\mathbf{u}}, \tilde{\varphi}, \tilde{\zeta}, \tilde{\theta}) \in \mathbf{V}_{KL}(\Omega) \times \mathbb{X}(\Omega) \times \mathbb{X}(\Omega) \times H^1(\Omega, \Gamma_0)$ are defined by:

$$\begin{aligned} \tilde{u}_\alpha(\tilde{x}, x_3) &= u_\alpha(\tilde{x}) - x_3\partial_\alpha u_3(\tilde{x}), \quad \mathbf{u}_H := (u_\alpha), \\ \tilde{u}_3(\tilde{x}, x_3) &= u_3(\tilde{x}), \\ \tilde{\theta}(\tilde{x}, x_3) &= \vartheta(\tilde{x}). \end{aligned} \tag{4.42}$$

We consider the case of a homogeneous material. The limit evolution problem decouples into two variational subproblems, namely, the flexural problem, which gives u_3, and the thermo-elastic membrane problem, which gives the couple $(\mathbf{u}_H, \vartheta)$. Moreover, we can characterize explicitly the limit magnetic potential $\tilde{\zeta}$ and the limit electric potential $\tilde{\varphi}$ as second order polynomial functions of x_3.

We choose test functions $\mathcal{V} = (\mathbf{0}, \psi, \xi, 0)$ in (4.41); after an integration by parts along x_3, and by the continuity of the applied magnetic and electric potential at the top and bottom faces, we obtain the characterization of the limit electric and magnetic potentials $\tilde{\varphi}$ and $\tilde{\zeta}$:

$$\tilde{\varphi}(\tilde{x}, x_3) = \sum_{k=0}^{2} f^k(\tilde{x}) x_3^k, \quad \tilde{\zeta}(\tilde{x}, x_3) = \sum_{k=0}^{2} z^k(\tilde{x}) x_3^k, \tag{4.43}$$

where

$$\begin{aligned} f^0 &= \langle \varphi \rangle + \frac{h^2}{2} \Lambda_{\alpha\beta} \partial_{\alpha\beta} u_3, \quad f^1 = \frac{[\![\varphi]\!]}{2h}, \quad f^2 = -\frac{1}{2} \Lambda_{\alpha\beta} \partial_{\alpha\beta} u_3, \\ z^0 &= \langle \zeta \rangle + \frac{h^2}{2} \Gamma_{\alpha\beta} \partial_{\alpha\beta} u_3, \quad z^1 = \frac{[\![\zeta]\!]}{2h}, \quad z^2 = -\frac{1}{2} \Gamma_{\alpha\beta} \partial_{\alpha\beta} u_3. \end{aligned}$$

with

$$\Lambda_{\alpha\beta} := \frac{\widetilde{M}^3_{33} \widetilde{P}^3_{3\alpha\beta} - \widetilde{\alpha}^3_{33} \widetilde{R}^3_{3\alpha\beta}}{\widetilde{M}^3_{33} \widetilde{X}^3_{33} - (\widetilde{\alpha}^3_{33})^2}, \quad \Gamma_{\alpha\beta} := \frac{\widetilde{\alpha}^3_{33} \widetilde{P}^3_{3\alpha\beta} - \widetilde{X}^3_{33} \widetilde{R}^3_{3\alpha\beta}}{\widetilde{M}^3_{33} \widetilde{X}^3_{33} - (\widetilde{\alpha}^3_{33})^2}.$$

The limit magnetic and electric potentials depend on the transversal displacement u_3 of the plate and on the values of the known boundary magnetic potentials $\zeta^{\pm}$ and electric potentials $\varphi^{\pm}$.

Thanks to the characterization (4.43) of the limit magnetic and electric potentials, we can now rewrite the limit evolution problem. After an integration by parts along x_3, we obtain a two-dimensional problem defined over the middle surface ω of the plate:

$$\left\{ \begin{array}{l} \text{Find } \widetilde{\mathcal{U}} = (\mathbf{u}_H(t), u_3(t), \vartheta(t)) \in \mathbf{V}_H(\omega, \gamma_0) \times V_3(\omega, \gamma_0) \times H^1(\omega, \gamma_0), \text{ such that} \\ 2h \displaystyle\int_\omega \left\{ \mathcal{N}^3_{\alpha\beta}(\mathbf{u}_H(t), \vartheta(t)) e_{\alpha\beta}(\mathbf{v}_H) + \rho \ddot{u}_\alpha(t) v_\alpha \right\} d\tilde{x} + \\ \quad +2h \displaystyle\int_\omega \left\{ \partial_t \widetilde{\mathcal{S}}^3(\mathbf{u}_H(t), \vartheta(t)) \eta - \widetilde{q}^1_\alpha(\vartheta(t)) \partial_\alpha \eta \right\} d\tilde{x} + \\ \quad + \dfrac{2h^3}{3} \displaystyle\int_\omega \left\{ \mathcal{M}^3_{\alpha\beta}(u_3(t)) \partial_{\alpha\beta} v_3 + \rho \partial_\alpha \ddot{u}_3(t) \partial_\alpha v_3 + \frac{3}{h^2} \rho \ddot{u}_3(t) v_3 \right\} d\tilde{x} = \mathcal{L}_3(\widetilde{\mathcal{V}}), \\ \text{for all } \widetilde{\mathcal{V}} = (\mathbf{v}_H, v_3, \eta) \in \mathbf{V}_H(\omega, \gamma_0) \times V_3(\omega, \gamma_0) \times H^1(\omega, \gamma_0), \ t \in (0, T), \end{array} \right.$$

with

$$\begin{aligned} \mathcal{L}_3(\widetilde{\mathcal{V}}) := & \int_\omega \left\{ \widetilde{f}_i v_i - m_\alpha \partial_\alpha v_3 + \widetilde{\rho}_e \psi + \widetilde{r} \eta \right\} d\tilde{x} + \int_{\gamma_1} \{ \widetilde{g}_i v_i - n_\alpha \partial_\alpha v_3 - \widetilde{\varrho} \eta \} \, d\gamma + \\ & - \int_\omega \left\{ \left(\widetilde{R}^3_{3\alpha\beta} [\![\zeta]\!] + \widetilde{P}^3_{3\alpha\beta} [\![\varphi]\!] \right) e_{\alpha\beta}(\mathbf{v}_H) - \left(\widetilde{p}^3_3 [\![\dot{\varphi}]\!] + \widetilde{m}^3_3 [\![\dot{\zeta}]\!] \right) \eta \right\} d\tilde{x}. \end{aligned}$$

The initial conditions are given by

$$\left\{ \begin{array}{l} \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0 = (u_{\alpha,0} - x_3 \partial_\alpha u_{3,0}, u_{3,0}), \\ \dot{\tilde{\mathbf{u}}}(0) = \tilde{\mathbf{u}}_1 = (u_{\alpha,1} - x_3 \partial_\alpha u_{3,1}, u_{3,1}), \\ \tilde{\theta}(0) = \tilde{\theta}_0 = \vartheta_0. \end{array} \right.$$

$(\mathcal{N}^3_{\alpha\beta})$, $(\mathcal{M}^3_{\alpha\beta})$, $\widetilde{S}^3$ and $(\widetilde{q}^1_\alpha)$ represent, respectively, the membrane stress tensor, the moment tensor, the reduced thermodynamic entropy and the reduced heat flow vector of the plate, whose components are defined by the constitutive laws below:

$$\left\{\begin{array}{l} \mathcal{N}^3_{\alpha\beta} := \widetilde{C}^3_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}_H) - \widetilde{\beta}^3_{\alpha\beta}\vartheta, \\ \widetilde{S}^3 := \widetilde{\beta}^3_{\alpha\beta} e_{\alpha\beta}(\mathbf{u}_H) + \widetilde{c}^3_v \vartheta, \\ \mathcal{M}^3_{\alpha\beta} := \widetilde{A}^3_{\alpha\beta\sigma\tau}\partial_{\sigma\tau} u_3, \\ \widetilde{q}^1_\alpha := -\frac{1}{T_0}\widetilde{K}^1_{\alpha\beta}\partial_\beta\vartheta, \end{array}\right. \tag{4.44}$$

where $\widetilde{A}^3_{\alpha\beta\sigma\tau} := \widetilde{C}^3_{\alpha\beta\sigma\tau} - \widetilde{P}^3_{3\alpha\beta}\Lambda_{\sigma\tau} - \widetilde{R}^3_{3\alpha\beta}\Gamma_{\sigma\tau}$.

The variational problem above can be split into two two-dimensional decoupled problems: namely, the flexural problem and the thermo-elastic membrane problem.

The flexural problem reads as follows:

$$\left\{\begin{array}{l} \text{Find } u_3(t) \in V_3(\omega,\gamma_0),\ t \in (0,T) \text{ such that} \\ \dfrac{2h^3}{3}\displaystyle\int_\omega \left\{\mathcal{M}^3_{\alpha\beta}(u_3(t))\partial_{\alpha\beta}v_3 + \rho\partial_\alpha\ddot{u}_3(t)\partial_\alpha v_3 + \frac{3}{h^2}\rho\ddot{u}_3(t)v_3\right\} d\tilde{x} = \\ \quad = \displaystyle\int_\omega \left\{\widetilde{f}_3 v_3 - m_\alpha\partial_\alpha v_3\right\} d\tilde{x} + \int_{\gamma_1}\{\widetilde{g}_3 v_3 - n_\alpha\partial_\alpha v_3\}\, d\gamma, \\ \text{for all } v_3 \in V_3(\omega,\gamma_0). \end{array}\right.$$

The two-dimensional thermo-elastic membrane problem takes the following form

$$\left\{\begin{array}{l} \text{Find } (\mathbf{u}_H(t),\vartheta(t)), \in \mathbf{V}_H(\omega,\gamma_0)\times H^1(\omega,\gamma_0),\ t\in(0,T) \text{ such that} \\ 2h\displaystyle\int_\omega \left\{\mathcal{N}^3_{\alpha\beta}(\mathbf{u}_H(t),\vartheta(t))e_{\alpha\beta}(\mathbf{v}_H) + \rho\ddot{u}_\alpha(t)v_\alpha\right\} d\tilde{x} + \\ \quad +2h\displaystyle\int_\omega \left\{\partial_t\widetilde{S}^3(\mathbf{u}_H(t),\vartheta(t))\eta - \widetilde{q}^1_\alpha(\vartheta(t))\partial_\alpha\eta\right\} d\tilde{x} = \\ \displaystyle\int_\omega \left\{\widetilde{f}_\alpha v_\alpha + \widetilde{\rho}_e\psi - \left(\widetilde{R}^3_{3\alpha\beta}[\![\zeta]\!] + \widetilde{P}^3_{3\alpha\beta}[\![\varphi]\!]\right) e_{\alpha\beta}(\mathbf{v}_H) + \left(\widetilde{r} + \widetilde{p}^3_3[\![\dot{\varphi}]\!] + \widetilde{m}^3_3[\![\dot{\zeta}]\!]\right)\eta\right\} d\tilde{x} + \\ \quad + \displaystyle\int_{\gamma_1}\{\widetilde{g}_\alpha v_\alpha - \widetilde{\varrho}\eta\}\, d\gamma, \\ \text{for all } (\mathbf{v}_H,\eta) \in \mathbf{V}_H(\omega,\gamma_0)\times H^1(\omega,\gamma_0). \end{array}\right.$$

By using Green's formulae on ω, we can derive the differential formulations of the above problems. The transversal displacement u_3 solves a flexural differential problem $(FDP)^3$ analogous to $(FDP)^1$ with $\mathcal{M}^1_{\alpha\beta}$ replaced by $\mathcal{M}^3_{\alpha\beta}$ defined in $(4.44)_3$.

The thermo-elastic state $(\mathbf{u}_H(t),\vartheta(t))$ solves the following thermo-elastic membrane differential problem:

$$\begin{cases} \textit{Field equations:} \\ 2h(\rho\ddot{u}_\alpha - \partial_\beta \mathcal{N}^3_{\alpha\beta}) = \widetilde{f}_\alpha + \widetilde{R}^3_{3\alpha\beta}[\![\partial_\beta\zeta]\!] + \widetilde{P}^3_{3\alpha\beta}[\![\partial_\beta\varphi]\!] & \text{in } \omega_T, \\ 2h(\partial_t\widetilde{\mathcal{S}}^3 + \partial_\alpha\widetilde{q}^1_\alpha) = \widetilde{r} + \widetilde{m}^3_3[\![\dot{\zeta}]\!] + \widetilde{p}^3_3[\![\dot{\varphi}]\!] & \text{in } \omega_T, \\ \textit{Initial conditions:} \\ u_\alpha(0) = u_{\alpha,0},\ \dot{u}_\alpha(0) = u_{\alpha,1},\ \vartheta(0) = \vartheta_0 & \text{in } \omega_T, \\ \textit{Boundary conditions:} \\ 2h\mathcal{N}^3_{\alpha\beta}\nu_\beta = \widetilde{g}_\alpha - \widetilde{R}^3_{3\alpha\beta}\nu_\beta[\![\zeta]\!] - \widetilde{P}^3_{3\alpha\beta}\nu_\beta[\![\varphi]\!] & \text{on } \gamma_1 \times (0,T), \\ 2h\widetilde{q}^1_\alpha\nu_\alpha = \widetilde{\varrho} & \text{on } \gamma_1 \times (0,T), \\ u_\alpha = \vartheta = 0 & \text{on } \gamma_0 \times (0,T). \end{cases}$$

As in the sensor-actuator problem, the piezomagnetic and piezoelectric behaviors are contained in the load terms on the right-hand side of the equations.

4.1.6.4 The Sensor Model

Theorem 4.9. *Under assumption (4.22), the sequence* $\{\mathcal{U}(\varepsilon)\}_{\varepsilon>0}$ *weakly converges to the limit* $\widetilde{\mathcal{U}} := (\tilde{\mathbf{u}}, \tilde{\varphi}, \tilde{\zeta}, \tilde{\theta})$ *in the space* $L^2(0,T;\mathbf{H}^1(\Omega)) \times L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$.

Proof. By the usual techniques, keeping in mind the different scalings on the electric and magnetic potentials, we can determine a uniform bound of the scaled energy $\{\mathcal{E}(\varepsilon)(t)\}_{\varepsilon>0}$. This result implies that the sequences $\{\boldsymbol{\kappa}(\varepsilon)\}_{\varepsilon>0}$, $\{\boldsymbol{\chi}(\varepsilon)\}_{\varepsilon>0}$, $\{\boldsymbol{\tau}(\varepsilon)\}_{\varepsilon>0}$ and $\{\boldsymbol{\gamma}(\varepsilon)\}_{\varepsilon>0}$ are bounded independently of ε in $L^2(0,T;\mathbf{L}^2(\Omega))$, meaning that $\kappa_{ij}(\varepsilon) \rightharpoonup \tilde{\kappa}_{ij}$, $\chi_i(\varepsilon) \rightharpoonup \tilde{\chi}_i$, $\tau_i(\varepsilon) \rightharpoonup \tilde{\tau}_i$ and $\gamma_i(\varepsilon) \rightharpoonup \tilde{\gamma}_i$ in $L^2(0,T;L^2(\Omega))$. Moreover, by means of Korn's and Poincaré's inequalities and from the definition of $\kappa_{ij}(\varepsilon)$, $\tau_i(\varepsilon)$, $\chi_i(\varepsilon)$ and $\gamma_i(\varepsilon)$, we infer that $\|\mathbf{u}(\varepsilon)\|_{1,\Omega}$, $\|\varphi(\varepsilon)\|_{1,\Omega}$, $\|\zeta(\varepsilon)\|_{1,\Omega}$ and $\|\theta(\varepsilon)\|_{1,\Omega}$ are also bounded, so that

$$\begin{aligned} \mathbf{u}(\varepsilon) &\rightharpoonup \tilde{\mathbf{u}} && \text{in } L^2(0,T;\mathbf{H}^1(\Omega)), \\ \dot{\mathbf{u}}(\varepsilon) &\rightharpoonup \dot{\tilde{\mathbf{u}}} && \text{in } L^2(0,T;\mathbf{L}^2(\Omega)), \\ \varphi(\varepsilon) &\rightharpoonup \tilde{\varphi} && \text{in } L^2(0,T;H^1(\Omega)), \\ \zeta(\varepsilon) &\rightharpoonup \tilde{\zeta} && \text{in } L^2(0,T;H^1(\Omega)), \\ \theta(\varepsilon) &\rightharpoonup \tilde{\theta} && \text{in } L^2(0,T;H^1(\Omega)). \end{aligned}$$

□

Theorem 4.10. *The weak limit* $\widetilde{\mathcal{U}}(t) = (\tilde{\mathbf{u}}(t), \tilde{\varphi}(t), \tilde{\zeta}(t), \tilde{\theta}(t))$ *is the solution to the limit variational problem:*

$$\begin{cases} \textit{Find } \widetilde{\mathcal{U}}(t) \in \mathbf{V}_{KL}(\Omega) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0),\ t \in (0,T) \\ \textit{such that } \widetilde{A}_4(\widetilde{\mathcal{U}}(t), \mathcal{V}) = \widetilde{L}_4(\mathcal{V}), \\ \textit{for all } \mathcal{V} \in \mathbf{V}_{KL}(\Omega) \times H^1(\omega,\gamma_0) \times H^1(\omega,\gamma_0) \times H^1(\omega,\gamma_0), \end{cases} \tag{4.45}$$

where

$$\widetilde{A}_4(\widetilde{\mathcal{U}}(t),\mathcal{V}) := \int_\Omega \Big\{ \Big(\widetilde{C}^4_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{P}^4_{\sigma\alpha\beta}\partial_\sigma\tilde{\varphi}(t) + \widetilde{R}^4_{\sigma\alpha\beta}\partial_\sigma\partial_\sigma\tilde{\zeta}(t) - \widetilde{\beta}^4_{\alpha\beta}\tilde{\theta}(t)\Big) e_{\alpha\beta}(\mathbf{v}) +$$
$$+ \Big(-\widetilde{P}^4_{\alpha\sigma\tau} e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{X}^4_{\alpha\beta}\partial_\beta\tilde{\varphi}(t) + \widetilde{\alpha}^4_{\alpha\beta}\partial_\beta\tilde{\zeta}(t) - \widetilde{p}^4_{\alpha}\tilde{\theta}(t)\Big)\partial_\alpha\psi +$$
$$+ \Big(-\widetilde{R}^4_{\alpha\sigma\tau} e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{\alpha}^4_{\alpha\beta}\partial_\beta\tilde{\varphi}(t) + \widetilde{M}^4_{\alpha\beta}\partial_\beta\tilde{\zeta}(t) - \widetilde{m}^4_{\alpha}\tilde{\theta}(t)\Big)\partial_\alpha\xi +$$
$$+ \Big(\widetilde{\beta}^4_{\alpha\beta} e_{\alpha\beta}(\dot{\tilde{\mathbf{u}}}(t)) - \widetilde{m}^4_{\alpha}\partial_\alpha\dot{\tilde{\zeta}}(t) - \widetilde{p}^4_{\alpha}\partial_\alpha\dot{\tilde{\varphi}}(t) + \widetilde{c}^4_v\dot{\tilde{\theta}}(t)\Big)\eta +$$
$$+ \widetilde{K}^1_{\alpha\beta}\partial_\beta\tilde{\theta}(t)\partial_\alpha\eta + \rho\ddot{\tilde{u}}_i(t)v_i \Big\}\, dx,$$
$$\widetilde{L}_4(\mathcal{V}) := (\mathbf{f},\mathbf{v}) + (\mathbf{g},\mathbf{v})_{\mathbf{L}^2(\hat{\Gamma})} + (r,\eta) - \tfrac{1}{T_0}(\varrho,\eta)_{L^2(\hat{\Gamma})} + (\rho_e,\psi) - (d,\psi)_{L^2(\hat{\Gamma})} - (b,\xi)_{L^2(\hat{\Gamma})}.$$

The reduced magneto-electro-thermo-elastic coefficients $\widetilde{C}^4_{\alpha\beta\sigma\tau}$, $\widetilde{X}^4_{\alpha\beta}$, $\widetilde{K}^1_{\alpha\beta}$, $\widetilde{P}^4_{\sigma\alpha\beta}$, $\widetilde{\beta}^4_{\alpha\beta}$, $\widetilde{p}^4_\alpha$ $\widetilde{m}^4_\alpha$, $\widetilde{R}^4_{\sigma\alpha\beta}$, $\widetilde{M}^4_{\alpha\beta}$, $\widetilde{\alpha}^4_{\alpha\beta}$ *and* $\widetilde{c}^4_v$ *are listed in Appendix 1.*

Proof. For convenience the proof is split into three parts, numbered from (i) to (iii).
(i) From definition of $\kappa_{ij}(\varepsilon)$, $\chi_i(\varepsilon)$, $\tau_i(\varepsilon)$ and $\gamma_i(\varepsilon)$, and thanks to the results of Theorem 4.3, there exists two constants C_M and C_K such that

$$\begin{array}{lll}
|e_{\alpha\beta}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant C_M e^{C_K T}, & |e_{\alpha 3}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}, & |e_{33}(\mathbf{u}(\varepsilon))|_{0,\Omega} \leqslant \varepsilon^2 C_M e^{C_K T}, \\
|\partial_\alpha\varphi(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T}, & |\partial_3\varphi(\varepsilon)|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}, & \\
|\partial_\alpha\zeta(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T}, & |\partial_3\zeta(\varepsilon)|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}, & \\
|\partial_\alpha\theta(\varepsilon)|_{0,\Omega} \leqslant C_M e^{C_K T}, & |\partial_3\theta(\varepsilon)|_{0,\Omega} \leqslant \varepsilon C_M e^{C_K T}. &
\end{array} \tag{4.46}$$

From the first set of inequalities $(4.46)_1$, we get that $e_{i3}(\mathbf{u}(\varepsilon)(t)) \to 0$ in $L^2(\Omega)$ for almost every $t \in (0,T)$. Also, as $\mathbf{u}(\varepsilon)(t) \rightharpoonup \tilde{\mathbf{u}}(t)$ in $\mathbf{H}^1(\Omega)$, we have that $e_{i3}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup e_{i3}(\tilde{\mathbf{u}}(t))$ and so $e_{i3}(\tilde{\mathbf{u}}(t)) = 0$. Thus, $\tilde{\mathbf{u}}(t) \in \mathbf{V}_{KL}(\Omega)$. Moreover, we obtain that $e_{\alpha\beta}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{\alpha\beta}(t) = e_{\alpha\beta}(\tilde{\mathbf{u}}(t))$ in $L^2(\Omega)$, $\frac{1}{\varepsilon}e_{\alpha 3}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{\alpha 3}(t)$ in $L^2(\Omega)$, $\frac{1}{\varepsilon}e_{33}(\mathbf{u}(\varepsilon)(t)) \to 0$ in $L^2(\Omega)$ and, also, $\frac{1}{\varepsilon^2}e_{33}(\mathbf{u}(\varepsilon)(t)) \rightharpoonup \kappa_{33}(t)$ in $L^2(\Omega)$.

From the last sets of inequalities $(4.46)_{2,3,4}$, since $\varphi(\varepsilon)(t) \rightharpoonup \tilde{\varphi}(t)$ in $H^1(\Omega)$, we infer that $\partial_\alpha\varphi(\varepsilon)(t) \rightharpoonup \tilde{\tau}_\alpha(t) = \partial_\alpha\tilde{\varphi}(t)$ in $L^2(\Omega)$ and, also, $\partial_3\varphi(\varepsilon)(t) \to 0$ in $L^2(\Omega)$, i.e., $\tilde{\varphi}(t) = \tilde{\varphi}(\tilde{x})(t)$ is independent of x_3. Besides, $\frac{1}{\varepsilon}\partial_3\varphi(\varepsilon)(t) \rightharpoonup \tau_3(t)$ in $L^2(\Omega)$. Analogously, since $\zeta(\varepsilon)(t) \rightharpoonup \tilde{\zeta}(t)$ in $H^1(\Omega)$, we obtain that $\partial_\alpha\zeta(\varepsilon)(t) \rightharpoonup \tilde{\chi}_\alpha(t) = \partial_\alpha\tilde{\zeta}(t)$ in $L^2(\Omega)$, $\partial_3\zeta(\varepsilon)(t) \to 0$ in $L^2(\Omega)$, i.e., $\tilde{\zeta}(t) = \tilde{\zeta}(\tilde{x})(t)$ and, finally, $\frac{1}{\varepsilon}\partial_3\zeta(\varepsilon)(t) \rightharpoonup \chi_3(t)$ in $L^2(\Omega)$. Similarly, since $\theta(\varepsilon)(t) \rightharpoonup \tilde{\theta}(t)$ in $H^1(\Omega)$, we obtain that $\partial_\alpha\theta(\varepsilon)(t) \rightharpoonup \tilde{\gamma}_\alpha(t) = \partial_\alpha\tilde{\theta}(t)$ in $L^2(\Omega)$, $\partial_3\theta(\varepsilon)(t) \to 0$ in $L^2(\Omega)$, i.e., $\tilde{\theta}(t) = \tilde{\theta}(\tilde{x})(t)$ and, finally, $\frac{1}{\varepsilon}\partial_3\theta(\varepsilon)(t) \rightharpoonup \gamma_3(t)$ in $L^2(\Omega)$.
(ii) *Computations of* κ_{i3}, τ_3, χ_3 *and* γ_3. Let us multiply problem (4.21) by ε^2 and let ε tend to zero. We get the following equation:

$$C_{3333}\kappa_{33} + 2C_{\alpha 333}\kappa_{\alpha 3} + P_{333}\tau_3 + R_{333}\chi_3 + C_{\alpha\beta 33}e_{\alpha\beta}(\tilde{\mathbf{u}}) + P_{\alpha 33}\partial_\alpha\tilde{\varphi} + R_{\alpha 33}\partial_\alpha\tilde{\zeta} - \beta_{33}\tilde{\theta} = 0.$$

By multiplying problem (4.21) by ε, we choose test functions $v_3 = \psi = \eta = \xi = 0$ and we let ε tend to zero, we have that

$$C_{\alpha 333}\kappa_{33} + 2C_{\alpha 3\beta 3}\kappa_{\beta 3} + P_{3\alpha 3}\tau_3 + R_{3\alpha 3}\chi_3 +$$
$$+ C_{\sigma\beta\alpha 3}e_{\sigma\beta}(\tilde{\mathbf{u}}) + P_{\sigma\alpha 3}\partial_\sigma\tilde{\varphi} + R_{\sigma\alpha 3}\partial_\sigma\tilde{\zeta} - \beta_{\alpha 3}\tilde{\theta} = 0.$$

Similarly, by multiplying problem (4.21) by ε and choosing test functions $v_i = \eta = \xi = 0$, when ε tends to zero, we find

$$-P_{333}\kappa_{33} - 2P_{3\alpha3}\kappa_{\alpha3} + X_{33}\tau_3 + \alpha_{33}\chi_3 - P_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}) + X_{\alpha3}\partial_\alpha\tilde{\varphi} + \alpha_{\alpha3}\partial_\alpha\tilde{\zeta} - p_3\tilde{\theta} = 0.$$

By multiplying problem (4.21) by ε and choosing test functions $v_i = \eta = \psi = 0$, when ε tends to zero, we find

$$-R_{333}\kappa_{33} - 2R_{3\alpha3}\kappa_{\alpha3} + M_{33}\chi_3 + \alpha_{\alpha3}\tau_3 - R_{3\alpha\beta}e_{\alpha\beta}(\tilde{\mathbf{u}}) + \alpha_{\alpha3}\partial_\alpha\tilde{\varphi} + M_{\alpha3}\partial_\alpha\tilde{\zeta} - m_3\tilde{\theta} = 0.$$

Finally, if we multiply by ε and choose test functions $v_i = \psi = \xi = 0$, we obtain the last equation

$$K_{33}\gamma_3 + K_{\alpha3}\partial_\alpha\tilde{\theta} = 0.$$

By solving the linear system constituted by the equations above, we are now in a position to characterize κ_{i3}, τ_3, χ_3 and γ_3. Let $\mathbf{l}^4 = (l_i^4)$ be the vector whose components are defined by $l_\alpha^4 := 2\kappa_{\alpha3}$ and $l_3^4 := \kappa_{33}$, then

$$\begin{aligned}
l_i^4 = &- \Big\{ \Big[d'_{ij}C_{\alpha\beta j3} + (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})P_{3\alpha\beta} + (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})R_{3\alpha\beta} \Big] e_{\alpha\beta}(\tilde{\mathbf{u}}) + \\
&+ \Big[d'_{ij}P_{\alpha j3} - (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})X_{\alpha3} - (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})\alpha_{\alpha3} \Big] \partial_\alpha\tilde{\varphi} + \\
&+ \Big[d'_{ij}R_{\alpha j3} - (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})\alpha_{\alpha3} - (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})M_{\alpha3} \Big] \partial_\alpha\tilde{\zeta} + \\
&+ \Big[-d'_{ij}\beta_{j3} - (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})p_3 - (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})m_3 \Big] \tilde{\theta} \Big\} \\
\tau_3 = &-c' \big\{ \big[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})C_{\alpha\beta i3} - M'_{33}P_{3\alpha\beta} + \alpha'_{33}R_{3\alpha\beta} \big] e_{\alpha\beta}(\tilde{\mathbf{u}}) + \\
&+ \big[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})P_{\alpha i3} + M'_{33}X_{\alpha3} - \alpha'_{33}\alpha_{\alpha3} \big] \partial_\alpha\tilde{\varphi} + \\
&+ \big[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})R_{\alpha i3} + M'_{33}\alpha_{\alpha3} - \alpha'_{33}M_{\alpha3} \big] \partial_\alpha\tilde{\zeta} + \\
&+ \big[-d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})\beta_{i3} + M'_{33}p_3 - \alpha'_{33}m_3 \big] \tilde{\theta} \big\} \\
\chi_3 = &-c' \big\{ \big[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})C_{\alpha\beta i3} - X'_{33}R_{3\alpha\beta} + \alpha'_{33}P_{3\alpha\beta} \big] e_{\alpha\beta}(\tilde{\mathbf{u}}) + \\
&+ \big[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})P_{\alpha i3} + X'_{33}M_{\alpha3} - \alpha'_{33}X_{\alpha3} \big] \partial_\alpha\tilde{\varphi} + \\
&+ \big[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})R_{\alpha i3} + X'_{33}M_{\alpha3} - \alpha'_{33}\alpha_{\alpha3} \big] \partial_\alpha\tilde{\zeta} + \\
&+ \big[-d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})\beta_{i3} + X'_{33}m_3 - \alpha'_{33}p_3 \big] \tilde{\theta} \big\} \\
\gamma_3 = &-K'_{\alpha3}\partial_\alpha\tilde{\theta}.
\end{aligned} \tag{4.47}$$

(iii) *Definition of the limit problem.* We let test functions be $\mathcal{V} = (\mathbf{v}, \psi, \xi, \eta) \in \mathbf{V}_{KL}(\Omega) \times H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0)$ in problem (4.21) and let $\varepsilon \to 0$; by substituting expressions (4.47), we obtain, as customary, the limit evolution problem

$$\begin{cases} \text{Find } \widetilde{\mathcal{U}}(t) \in \mathbf{V}_{KL}(\Omega) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0),\ t \in (0,T) \\ \text{such that} \\ \int_\Omega \Big\{ \Big(\widetilde{C}^4_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{P}^4_{\sigma\alpha\beta}\partial_\sigma\tilde{\varphi}(t) + \widetilde{R}^4_{\sigma\alpha\beta}\partial_\sigma\tilde{\zeta}(t) - \widetilde{\beta}^4_{\alpha\beta}\tilde{\theta}(t)\Big) e_{\alpha\beta}(\mathbf{v}) + \\ \quad + \Big(-\widetilde{P}^4_{\alpha\sigma\tau} e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{X}^4_{\alpha\beta}\partial_\beta\tilde{\varphi}(t) + \widetilde{\alpha}^4_{\alpha\beta}\partial_\beta\tilde{\zeta}(t) - \widetilde{p}^4_\alpha\tilde{\theta}(t)\Big)\partial_\alpha\psi + \\ \quad + \Big(-\widetilde{R}^4_{\alpha\sigma\tau} e_{\sigma\tau}(\tilde{\mathbf{u}}(t)) + \widetilde{\alpha}^4_{\alpha\beta}\partial_\beta\tilde{\varphi}(t) + \widetilde{M}^4_{\alpha\beta}\partial_\beta\tilde{\zeta}(t) - \widetilde{m}^4_\alpha\tilde{\theta}(t)\Big)\partial_\alpha\xi + \\ \quad + \Big(\widetilde{\beta}^4_{\alpha\beta} e_{\alpha\beta}(\dot{\tilde{\mathbf{u}}}(t)) - \widetilde{m}^4_\alpha\partial_\alpha\dot{\tilde{\zeta}}(t) - \widetilde{p}^4_\alpha\partial_\alpha\dot{\tilde{\varphi}}(t) + \widetilde{c}^4_v\dot{\tilde{\theta}}(t)\Big)\eta + \\ \quad + \widetilde{K}^1_{\alpha\beta}\partial_\beta\tilde{\theta}(t)\partial_\alpha\eta + \rho\ddot{\tilde{u}}_i(t)v_i\Big\}\, dx = \widetilde{L}_4(\mathcal{V}), \\ \text{for all } \mathcal{V} \in \mathbf{V}_{KL}(\Omega) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0). \end{cases}$$

Thus the main result is achieved. □

The limit evolution problem

In this section we present the variational and differential formulations of the evolution problem for a sensor magneto-electro-thermo-elastic plate. The primary unknowns $\widetilde{\mathcal{U}}(t) = (\tilde{\mathbf{u}}(t), \tilde{\varphi}(t), \tilde{\zeta}(t), \tilde{\theta}(t)) \in \mathbf{V}_{KL}(\Omega) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0) \times H^1(\Omega,\Gamma_0)$ satisfy the following kinematical assumptions

$$\begin{array}{l} \tilde{u}_\alpha(\tilde{x}, x_3) = u_\alpha(\tilde{x}) - x_3\partial_\alpha u_3(\tilde{x}),\ \ \mathbf{u}_H := (u_\alpha), \\ \tilde{u}_3(\tilde{x}, x_3) = u_3(\tilde{x}), \\ \tilde{\varphi}(\tilde{x}, x_3) = \phi(\tilde{x}), \\ \tilde{\zeta}(\tilde{x}, x_3) = \varsigma(\tilde{x}), \\ \tilde{\theta}(\tilde{x}, x_3) = \vartheta(\tilde{x}). \end{array} \tag{4.48}$$

The limit evolution problem for a homogeneous material decouples into two variational subproblems, namely, the flexural problem, which gives u_3, and the magneto-electro-thermo-elastic membrane problem, which gives the quadruplet $(\mathbf{u}_H, \phi, \varsigma, \vartheta)$. After an integration by parts along x_3, we obtain a two-dimensional problem defined over the middle surface ω of the plate:

$$\begin{cases} \text{Find } (\mathbf{u}_H(t), u_3(t), \phi(t), \varsigma(t), \vartheta(t)) \in \mathbf{V}_H(\omega,\gamma_0) \times V_3(\omega,\gamma_0) \times [H^1(\omega,\gamma_0)]^3, \\ t \in (0,T) \text{ such that} \\ 2h\int_\omega \Big\{ \mathcal{N}^4_{\alpha\beta}(\mathbf{u}_H(t), \phi(t), \varsigma(t), \vartheta(t)) e_{\alpha\beta}(\mathbf{v}_H) + \rho\ddot{u}_\alpha(t)v_\alpha\Big\}\, d\tilde{x} + \\ \ -2h\int_\omega \Big\{ \widetilde{D}^4_\alpha(\mathbf{u}_H(t), \phi(t), \varsigma(t), \vartheta(t))\partial_\alpha\psi + \widetilde{B}^4_\alpha(\mathbf{u}_H(t), \phi(t), \varsigma(t), \vartheta(t))\partial_\alpha\xi\Big\}\, d\tilde{x} + \\ \ +2h\int_\omega \Big\{ \partial_t\widetilde{S}^4(\mathbf{u}_H(t), \phi(t), \varsigma(t), \vartheta(t))\eta - \widetilde{q}^1_\alpha(\vartheta(t))\partial_\alpha\eta\Big\}\, d\tilde{x} + \\ \ +\dfrac{2h^3}{3}\int_\omega \Big\{ \mathcal{M}^3_{\alpha\beta}(u_3(t))\partial_{\alpha\beta}v_3 + \rho\partial_\alpha\ddot{u}_3(t)\partial_\alpha v_3 + \dfrac{3}{h^2}\rho\ddot{u}_3(t)v_3\Big\}\, d\tilde{x} = \mathcal{L}_4(\widetilde{\mathcal{V}}), \\ \text{for all } \widetilde{\mathcal{V}} = (\mathbf{v}_H, v_3, \psi, \xi, \eta) \in \mathbf{V}_H(\omega,\gamma_0) \times V_3(\omega,\gamma_0) \times [H^1(\omega,\gamma_0)]^3,\ t \in (0,T), \end{cases}$$

with

$$\mathcal{L}_4(\widetilde{\mathcal{V}}) := \int_\omega \left\{ \widetilde{f}_i v_i - m_\alpha \partial_\alpha v_3 + \widetilde{\rho}_e \psi + \widetilde{r}\eta \right\} d\tilde{x} + \int_{\gamma_1} \left\{ \widetilde{g}_i v_i - n_\alpha \partial_\alpha v_3 - \widetilde{d}\psi - \widetilde{b}\xi - \widetilde{\varrho}\eta \right\} d\gamma.$$

The initial conditions are given by

$$\begin{cases} \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0 = (u_{\alpha,0} - x_3 \partial_\alpha u_{3,0}, u_{3,0}), \\ \dot{\tilde{\mathbf{u}}}(0) = \tilde{\mathbf{u}}_1 = (u_{\alpha,1} - x_3 \partial_\alpha u_{3,1}, u_{3,1}), \\ \tilde{\theta}(0) = \tilde{\theta}_0 = \vartheta_0. \end{cases}$$

$(\mathcal{N}^4_{\alpha\beta})$, $(\mathcal{M}^4_{\alpha\beta})$, $(\widetilde{D}^4_\alpha)$, $(\widetilde{B}^4_\alpha)$, $\widetilde{\mathcal{S}}^4$ and $(\widetilde{q}^1_\alpha)$ represent, respectively, the membrane stress tensor, the moment tensor, the reduced electric displacement vector, the reduced magnetic induction vector, the reduced thermodynamic entropy and the reduced heat flow vector of the plate, whose components are defined by the constitutive laws below:

$$\begin{cases} \mathcal{N}^4_{\alpha\beta} := \widetilde{C}^4_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}_H) + \widetilde{P}^4_{\sigma\alpha\beta} \partial_\sigma \phi + \widetilde{R}^4_{\sigma\alpha\beta} \partial_\sigma \varsigma - \widetilde{\beta}^4_{\alpha\beta} \vartheta, \\ \widetilde{D}^4_\alpha := \widetilde{P}^4_{\alpha\sigma\tau} e_{\sigma\tau}(\mathbf{u}_H) - \widetilde{X}^4_{\alpha\beta} \partial_\beta \phi - \alpha^4_{\alpha\beta} \partial_\beta \varsigma + \widetilde{p}^4_\alpha \vartheta, \\ \widetilde{B}^4_\alpha := \widetilde{R}^4_{\alpha\sigma\tau} e_{\sigma\tau}(\mathbf{u}_H) - \widetilde{\alpha}^4_{\alpha\beta} \partial_\beta \phi - M^4_{\alpha\beta} \partial_\beta \varsigma + \widetilde{m}^4_\alpha \vartheta, \\ \widetilde{\mathcal{S}}^4 := \widetilde{\beta}^4_{\alpha\beta} e_{\alpha\beta}(\mathbf{u}_H) - \widetilde{p}^4_\alpha \partial_\alpha \phi - \widetilde{m}^4_\alpha \partial_\alpha \varsigma + \widetilde{c}^4_v \vartheta, \\ \mathcal{M}^4_{\alpha\beta} := \widetilde{C}^4_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} u_3, \\ \widetilde{q}^1_\alpha := -\frac{1}{T_0} \widetilde{K}^1_{\alpha\beta} \partial_\beta \vartheta, \end{cases} \tag{4.49}$$

The variational problem above can be split into two two-dimensional decoupled problems: namely, the flexural problem and the magneto-electro-thermo-elastic membrane problem.

The flexural problem reads as follows:

$$\begin{cases} \text{Find } u_3(t) \in V_3(\omega, \gamma_0),\ t \in (0, T) \text{ such that} \\ \dfrac{2h^3}{3} \displaystyle\int_\omega \left\{ \mathcal{M}^4_{\alpha\beta}(u_3(t)) \partial_{\alpha\beta} v_3 + \rho \partial_\alpha \ddot{u}_3(t) \partial_\alpha v_3 + \frac{3}{h^2} \rho \ddot{u}_3(t) v_3 \right\} d\tilde{x} = \\ \quad = \displaystyle\int_\omega \left\{ \widetilde{f}_3 v_3 - m_\alpha \partial_\alpha v_3 \right\} d\tilde{x} + \int_{\gamma_1} \{ \widetilde{g}_3 v_3 - n_\alpha \partial_\alpha v_3 \} d\gamma, \\ \text{for all } v_3 \in V_3(\omega, \gamma_0). \end{cases}$$

The two-dimensional magneto-electro-thermo-elastic membrane problem takes the following form

$$\begin{cases} \text{Find } (\mathbf{u}_H(t), \phi(t), \varsigma(t), \vartheta(t)), \in \mathbf{V}_H(\omega, \gamma_0) \times [H^1(\omega, \gamma_0)]^3,\ t \in (0, T) \text{ such that} \\ 2h \displaystyle\int_\omega \left\{ \mathcal{N}^4_{\alpha\beta}(\mathbf{u}_H(t), \phi(t), \varsigma(t), \vartheta(t)) e_{\alpha\beta}(\mathbf{v}_H) + \rho \ddot{u}_\alpha(t) v_\alpha \right\} d\tilde{x} + \\ \quad -2h \displaystyle\int_\omega \left\{ \widetilde{D}^4_\alpha(\mathbf{u}_H(t), \phi(t), \varsigma(t), \vartheta(t)) \partial_\alpha \psi + \widetilde{B}^4_\alpha(\mathbf{u}_H(t), \phi(t), \varsigma(t), \vartheta(t)) \partial_\alpha \xi \right\} d\tilde{x} + \\ \quad +2h \displaystyle\int_\omega \left\{ \partial_t \widetilde{\mathcal{S}}^4(\mathbf{u}_H(t), \phi(t), \varsigma(t), \vartheta(t)) \eta - \widetilde{q}^1_\alpha(\vartheta(t)) \partial_\alpha \eta \right\} d\tilde{x} = \\ \quad = \displaystyle\int_\omega \left\{ \widetilde{f}_\alpha v_\alpha + \widetilde{\rho}_e \psi + \widetilde{r}\eta \right\} d\tilde{x} + \int_{\gamma_1} \left\{ \widetilde{g}_\alpha v_\alpha - \widetilde{d}\psi - \widetilde{b}\xi - \widetilde{\varrho}\eta \right\} d\gamma, \\ \text{for all } (\mathbf{v}_H, \psi, \xi, \eta) \in \mathbf{V}_H(\omega, \gamma_0) \times [H^1(\omega, \gamma_0)]^3. \end{cases}$$

By means of Green's formulae on ω, we have that the transversal displacement u_3 solves a flexural differential problem $(FDP)^4$ analogous to $(FDP)^1$ with $\mathcal{M}^1_{\alpha\beta}$ replaced by $\mathcal{M}^4_{\alpha\beta}$ defined in $(4.49)_5$, whereas the membrane magneto-electro-thermo-elastic state $(\mathbf{u}_H(t), \phi(t), \varsigma(t), \vartheta(t))$ solves the following magneto-electro-thermo-elastic membrane differential problem:

$$\begin{cases} \textit{Field equations:} & \\ 2h(\rho\ddot{u}_\alpha - \partial_\beta \mathcal{N}^4_{\alpha\beta}) = \widetilde{f}_\alpha & \text{in } \omega_T, \\ 2h\partial_\alpha \widetilde{D}^4_\alpha = \widetilde{\rho}_e & \text{in } \omega_T, \\ 2h\partial_\alpha \widetilde{B}^4_\alpha = -(b^+ + b^-) & \text{in } \omega_T, \\ 2h(\partial_t \widetilde{S}^4 + \partial_\alpha \widetilde{q}^1_\alpha) = \widetilde{r} & \text{in } \omega_T, \\ \textit{Initial conditions:} & \\ u_\alpha(0) = u_{\alpha,0},\ \dot{u}_\alpha(0) = u_{\alpha,1},\ \vartheta(0) = \vartheta_0 & \text{in } \omega_T, \\ \textit{Boundary conditions:} & \\ 2h\mathcal{N}^4_{\alpha\beta}\nu_\beta = \widetilde{g}_\alpha & \text{on } \gamma_1 \times (0,T), \\ 2h\widetilde{D}^4_\alpha \nu_\alpha = \widetilde{d} & \text{on } \gamma_1 \times (0,T), \\ 2h\widetilde{B}^4_\alpha \nu_\alpha = \widetilde{b} & \text{on } \gamma_1 \times (0,T), \\ 2h\widetilde{q}^1_\alpha \nu_\alpha = \widetilde{\varrho} & \text{on } \gamma_1 \times (0,T), \\ u_\alpha = \phi = \varsigma = \vartheta = 0 & \text{on } \gamma_0 \times (0,T). \end{cases} \tag{4.50}$$

Remark 4.2. Equation $(4.50)_3$ deserves some comments. Indeed, one would expect a two-dimensional counterpart of Gauss' law $\operatorname{div}\mathbf{B} = 0$ in three dimensions; however, $(4.50)_3$ involves a source term. This is due to the fact that b^+ and b^- represent applied magnetic inductions whose direction is *orthogonal* to the plane containing the middle surface of the plate, whereas the divergence $\partial_\alpha \widetilde{B}^4_\alpha$ is taken with respect to the in-plane coordinates. An analogous situation is that of equation $(4.37)_2$ of the membrane thermo-piezomagnetic problem arising in the actuator-sensor model, where the source term is given by the electric potentials applied on the upper and lower faces of the plate, due to the magneto-electric coupling.

Conclusions

We set forth an asymptotic two-dimensional plate model for linear magneto-electro-thermo-elastic sensors and actuators, under the hypotheses of anisotropy and homogeneity. A validation of the results obtained by the asymptotic expansion method is accomplished thanks to weak convergence theorems 4.3, 4.5, 4.7 and 4.9. Each of the four plate problems originating from the four different boundary conditions presented decouples into a flexural problem and a partially or totally coupled membrane problem, depending on how the plate is supposed to behave. As in [63], the four models differ between one another in the scaling assumptions on the electric and magnetic potentials (see (4.12), (4.16), (4.18), (4.20)). Although the formulae in Appendix 1 giving the reduced material coefficients may appear complex, numerical values of such coefficients can be easily computed, as reported in Appendix 2 for the usual $BaTiO_3$-$CoFe_2O_4$ composite.

Concerning directions for future research, a first important issue to deal with is, of course, coming up with efficient numerical methods to perform simulations, based either on the two-dimensional plate problems or on the three-dimensional problem. Then, another problem of interest is to extend our results to shell-like bodies; in situations where the geometry is particularly simple, as in the case of cylindrical shells, one should probably be able to determine analytical solutions by separation of variables, as in [62]. Finally, a further interesting problem is the study of a whole laminated structure (plate-like or shell-like) including a thin magneto-electro-thermo-elastic layer (see, e.g., [59]).

Appendix 1

In the sequel we define all the sets of reduced magneto-electro-thermo-elastic coefficients characterizing the four different models. We recall that, in each of the definitions below and in the following subsections, $(d_{ij}) := (C_{i3j3})^{-1}$, i.e., (d_{ij}) is the inverse of a second-order tensor whose ij components are C_{i3j3}. Also, the reduced thermal conductivity tensor $(\widetilde{K}^1_{\alpha\beta})$ is the same throughout the four models, thus we give its definition once and for all in the list below, along with the definitions of general reduced coefficients needed in the following subsections.

$$
\begin{aligned}
C'_{\alpha\beta\sigma\tau} &:= C_{\alpha\beta\sigma\tau} - d_{ij}C_{\alpha\beta i3}C_{\sigma\tau j3}, & R'_{3\alpha\beta} &:= R_{3\alpha\beta} - d_{ij}C_{\alpha\beta i3}R_{3j3},\\
X'_{p3} &:= X_{p3} + d_{ij}P_{3i3}P_{pj3}, & \beta'_{\alpha\beta} &:= \beta_{\alpha\beta} - d_{ij}C_{\alpha\beta i3}\beta_{j3},\\
M'_{p3} &:= M_{p3} + d_{ij}R_{3i3}R_{pj3}, & p'_3 &:= p_3 + d_{ij}P_{3j3}\beta_{i3},\\
\alpha'_{33} &:= \alpha_{33} + d_{ij}P_{3i3}R_{3j3}, & m'_3 &:= m_3 + d_{ij}R_{3i3}\beta_{j3},\\
P'_{p\alpha\beta} &:= P_{p\alpha\beta} - d_{ij}P_{pi3}C_{\alpha\beta j3}, & c'_v &:= c_v + d_{ij}\beta_{i3}\beta_{j3},
\end{aligned}
$$

$$
\begin{aligned}
K'_{\alpha 3} &:= \frac{K_{\alpha 3}}{K_{33}}, & \ell' &:= \frac{1}{M'_{33}},\\
\widetilde{K}^1_{\alpha\beta} &:= K_{\alpha\beta} - K'_{\alpha 3}K_{\beta 3}, & c' &:= \frac{1}{M'_{33}X'_{33} - (\alpha'_{33})^2},\\
k' &:= \frac{1}{X'_{33}}, & d'_{ij} &:= d_{ij} - c'd_{i\ell}\left(M'_{33}P_{3\ell 3}P_{3j3} + X'_{33}R_{3\ell 3}R_{3j3} + \right.\\
& & &\qquad \left. -\alpha'_{33}\left(R_{3\ell 3}P_{3j3} + P_{3\ell 3}R_{3j3}\right)\right).
\end{aligned}
$$

Reduced Coefficients for the Sensor-Actuator Model

$$
\begin{aligned}
\widetilde{C}^1_{\alpha\beta\sigma\tau} &:= C_{\alpha\beta\sigma\tau} - C_{\alpha\beta i3}d_{ij}\left(C_{\sigma\tau j3} + k'P_{3j3}P'_{3\sigma\tau}\right) + k'P_{3\alpha\beta}P'_{3\sigma\tau},\\
\widetilde{P}^1_{\sigma\alpha\beta} &:= P_{\sigma\alpha\beta} - C_{\alpha\beta i3}d_{ij}\left(P_{\sigma j3} - k'P_{3j3}X'_{\sigma 3}\right) - k'P_{3\alpha\beta}X'_{\sigma 3},\\
\widetilde{R}^1_{3\alpha\beta} &:= R_{3\alpha\beta} - C_{\alpha\beta i3}d_{ij}\left(R_{3j3} - k'P_{3j3}\alpha'_{33}\right) - k'P_{3\alpha\beta}\alpha'_{33},\\
\widetilde{\beta}^1_{\alpha\beta} &:= \beta_{\alpha\beta} - C_{\alpha\beta i3}d_{ij}\left(\beta_{j3} - k'P_{3j3}p'_3\right) - k'P_{3\alpha\beta}p'_3,\\
\widetilde{X}^1_{\alpha\beta} &:= X_{\alpha\beta} + P_{\alpha i3}d_{ij}\left(P_{\beta j3} - k'P_{3j3}X'_{\beta 3}\right) - k'X_{\alpha 3}X'_{\beta 3},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\alpha}^1_{\alpha 3} &:= \alpha_{\alpha 3} + P_{\alpha i3}d_{ij}\left(R_{3j3} - k'P_{3j3}\alpha'_{33}\right) - k'X_{\alpha 3}\alpha'_{33},\\
\widetilde{p}^1_{\alpha} &:= p_{\alpha} + P_{\alpha i3}d_{ij}\left(\beta_{j3} - k'P_{3j3}p'_3\right) - k'X_{\alpha 3}p'_3,\\
\widetilde{M}^1_{33} &:= M_{33} + R_{3i3}d_{ij}\left(R_{3j3} - k'P_{3j3}\alpha'_{33}\right) - k'\alpha_{33}\alpha'_{33},\\
\widetilde{m}^1_3 &:= m_3 + R_{3i3}d_{ij}\left(\beta_{j3} - k'P_{3j3}p'_3\right) - k'\alpha_{33}p'3,\\
\widetilde{c}^1_v &:= c_v + \beta_{i3}d_{ij}\left(\beta_{j3} - k'P_{3j3}p'_3\right) - k'p_3p'_3.
\end{aligned}
$$

Reduced Coefficients for the Actuator-Sensor Model

$$
\begin{aligned}
\widetilde{C}^2_{\alpha\beta\sigma\tau} &\coloneqq C_{\alpha\beta\sigma\tau} - C_{\alpha\beta i3} d_{ij} \left(C_{\sigma\tau j3} + \ell' R_{3j3} R'_{3\sigma\tau} \right) + \ell' R_{3\alpha\beta} R'_{3\sigma\tau},\\
\widetilde{R}^2_{\sigma\alpha\beta} &\coloneqq R_{\sigma\alpha\beta} - C_{\alpha\beta i3} d_{ij} \left(R_{\sigma j3} - \ell' R_{3j3} M'_{\sigma 3} \right) - \ell' R_{3\alpha\beta} M'_{\sigma 3},\\
\widetilde{P}^2_{3\alpha\beta} &\coloneqq P_{3\alpha\beta} - C_{\alpha\beta i3} d_{ij} \left(P_{3j3} - \ell' R_{3j3} \alpha'_{33} \right) - \ell' R_{3\alpha\beta} \alpha'_{33},\\
\widetilde{\beta}^2_{\alpha\beta} &\coloneqq \beta_{\alpha\beta} - C_{\alpha\beta i3} d_{ij} \left(\beta_{j3} - \ell' R_{3j3} p'_3 \right) - \ell' R_{3\alpha\beta} m'_3,\\
\widetilde{M}^2_{\alpha\beta} &\coloneqq M_{\alpha\beta} + R_{\alpha i3} d_{ij} \left(R_{\beta j3} - \ell' R_{3j3} M'_{\beta 3} \right) - \ell' M_{\alpha 3} M'_{\beta 3},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\alpha}^2_{\alpha 3} &\coloneqq \alpha_{\alpha 3} + R_{\alpha i3} d_{ij} \left(P_{3j3} - \ell' R_{3j3} \alpha'_{33} \right) - \ell' M_{\alpha 3} \alpha'_{33},\\
\widetilde{m}^2_{\alpha} &\coloneqq m_{\alpha} + R_{\alpha i3} d_{ij} \left(\beta_{j3} - \ell' R_{3j3} m'_3 \right) - \ell' M_{\alpha 3} m'_3,\\
\widetilde{X}^2_{33} &\coloneqq X_{33} + P_{3i3} d_{ij} \left(P_{3j3} - \ell' R_{3j3} \alpha'_{33} \right) - \ell' \alpha_{33} \alpha'_{33},\\
\widetilde{p}^2_{3} &\coloneqq p_3 + P_{3i3} d_{ij} \left(\beta_{j3} - \ell' R_{3j3} m'_3 \right) - \ell' \alpha_{33} m' 3,\\
\widetilde{c}^2_{v} &\coloneqq c_v + \beta_{i3} d_{ij} \left(\beta_{j3} - \ell' R_{3j3} m'_3 \right) - \ell' m_3 m'_3.
\end{aligned}
$$

Reduced Coefficients for the Actuator Model

$$
\begin{aligned}
\widetilde{C}^3_{\alpha\beta\sigma\tau} &= C'_{\alpha\beta\sigma\tau} = C_{\alpha\beta\sigma\tau} - d_{ij} C_{\alpha\beta i3} C_{\sigma\tau j3},\\
\widetilde{X}^3_{p3} &= X'_{p3} = X_{p3} + d_{ij} P_{3i3} P_{pj3},\\
\widetilde{M}^3_{p3} &= M'_{p3} = M_{p3} + d_{ij} R_{3i3} R_{pj3},\\
\widetilde{\alpha}^3_{33} &= \alpha'_{33} = \alpha_{33} + d_{ij} P_{3i3} R_{3j3},\\
\widetilde{P}^3_{p\alpha\beta} &= P'_{p\alpha\beta} = P_{p\alpha\beta} - d_{ij} P_{pi3} C_{\alpha\beta j3},\\
\widetilde{R}^3_{3\alpha\beta} &= R'_{3\alpha\beta} = R_{3\alpha\beta} - d_{ij} C_{\alpha\beta i3} R_{3j3},\\
\widetilde{\beta}^3_{\alpha\beta} &= \beta'_{\alpha\beta} = \beta_{\alpha\beta} - d_{ij} C_{\alpha\beta i3} \beta_{j3},\\
\widetilde{p}^3_{3} &= p'_3 = p_3 + d_{ij} P_{3j3} \beta_{i3},\\
\widetilde{m}^3_{3} &= m'_3 = m_3 + d_{ij} R_{3i3} \beta_{j3},\\
\widetilde{c}^3_{v} &= c'_v = c_v + d_{ij} \beta_{i3} \beta_{j3}.
\end{aligned}
$$

Reduced Coefficients for the Sensor Model

$$\begin{aligned}\widetilde{C}^4_{\alpha\beta\sigma\tau} \coloneqq\ & C_{\alpha\beta\sigma\tau} - C_{\alpha\beta i3}\Big[d'_{ij}C_{\sigma\tau j3} + (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})P_{3\sigma\tau} + \\ & +(\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})R_{3\sigma\tau}\Big] - c'P_{3\alpha\beta}\left[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})C_{\alpha\beta i3} + \right. \\ & \left. -M'_{33}P_{3\alpha\beta} + \alpha'_{33}R_{3\alpha\beta}\right] - c'R_{3\alpha\beta}\left[d_{ij}(X'_{33}R_{3j3} + \right. \\ & \left. -\alpha'_{33}P_{3j3})C_{\alpha\beta i3} - X'_{33}R_{3\alpha\beta} + \alpha'_{33}P_{3\alpha\beta}\right],\end{aligned}$$

$$\begin{aligned}\widetilde{P}^4_{\sigma\alpha\beta} \coloneqq\ & P_{\sigma\alpha\beta} - C_{\alpha\beta i3}\Big[d'_{ij}P_{\sigma j3} - (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})X_{\sigma 3} - (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})\alpha_{\sigma 3}\Big] + \\ & -c'P_{3\alpha\beta}\left[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})P_{\sigma i3} + M'_{33}X_{\sigma 3} - \alpha'_{33}\alpha_{\sigma 3}\right] + \\ & -c'R_{3\alpha\beta}\left[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})P_{\sigma i3} + X'_{33}M_{\sigma 3} - \alpha'_{33}X_{\sigma 3}\right],\end{aligned}$$

$$\begin{aligned}\widetilde{R}^4_{\sigma\alpha\beta} \coloneqq\ & R_{\sigma\alpha\beta} - C_{\alpha\beta i3}\Big[d'_{ij}R_{\sigma j3} - (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})\alpha_{\sigma 3} - (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})M_{\sigma 3}\Big] + \\ & -c'P_{3\alpha\beta}\left[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})R_{\sigma i3} + M'_{33}\alpha_{\sigma 3} - \alpha'_{33}M_{\sigma 3}\right] + \\ & -c'R_{3\alpha\beta}\left[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})R_{\sigma i3} + X'_{33}M_{\sigma 3} - \alpha'_{33}\alpha_{\sigma 3}\right],\end{aligned}$$

$$\begin{aligned}\widetilde{\beta}^4_{\alpha\beta} \coloneqq\ & \beta_{\alpha\beta} - C_{\alpha\beta i3}\Big[d'_{ij}\beta_{j3} + (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})p_3 + (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})m_3\Big] + \\ & -c'P_{3\alpha\beta}\left[d_{ij}(M'_{33}P_{3j3} + \alpha'_{33}R_{3j3})\beta_{i3} - M'_{33}p_3 + \alpha'_{33}m_3\right] + \\ & -c'R_{3\alpha\beta}\left[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})\beta_{i3} - X'_{33}m_3 + \alpha'_{33}p_3\right],\end{aligned}$$

$$\begin{aligned}\widetilde{X}^4_{\alpha\beta} \coloneqq\ & X_{\alpha\beta} + P_{\alpha i3}\Big[d'_{ij}P_{\beta j3} - (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})X_{\beta 3} - (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})\alpha_{\beta 3}\Big] + \\ & -c'X_{\alpha 3}\left[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})P_{\beta i3} + M'_{33}X_{\beta 3} - \alpha'_{33}\alpha_{\beta 3}\right] + \\ & -c'\alpha_{\alpha 3}\left[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})P_{\beta i3} + X'_{33}M_{\beta 3} - \alpha'_{33}X_{\beta 3}\right],\end{aligned}$$

$$\begin{aligned}\widetilde{\alpha}^4_{\alpha\beta} \coloneqq\ & \alpha_{\alpha\beta} + P_{\alpha i3}\Big[d'_{ij}R_{\beta j3} - (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})\alpha_{\beta 3} - (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})M_{\beta 3}\Big] + \\ & -c'X_{\alpha 3}\Big[d'_{ij}R_{\beta j3} - (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})\alpha_{\beta 3} - (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})M_{\beta 3}\Big] + \\ & -c'\alpha_{\alpha 3}\left[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})R_{\alpha i3} + X'_{33}M_{\alpha 3} - \alpha'_{33}\alpha_{\alpha 3}\right],\end{aligned}$$

$$\begin{aligned}\widetilde{p}^4_{\alpha} \coloneqq\ & p_{\alpha} - P_{\alpha i3}\Big[d'_{ij}\beta_{j3} + (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})p_3 + (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})m_3\Big] + \\ & -c'X_{\alpha 3}\left[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})\beta_{i3} - M'_{33}p_3 + \alpha'_{33}m_3\right] + \\ & -c'\alpha_{\alpha 3}\left[d_{ij}(X'_{33}R_{3j3} + \alpha'_{33}P_{3j3})\beta_{i3} - X'_{33}m_3 + \alpha'_{33}p_3\right],\end{aligned}$$

$$\begin{aligned}\widetilde{M}^4_{\alpha\beta} \coloneqq\ & M_{\alpha\beta} + R_{\alpha i3}\Big[d'_{ij}R_{\beta j3} - (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})\alpha_{\beta 3} - (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})M_{\beta 3}\Big] + \\ & -c'\alpha_{\alpha 3}\left[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})R_{\beta i3} + M'_{33}\alpha_{\beta 3} - \alpha'_{33}M_{\beta 3}\right] + \\ & -c'M_{\alpha 3}\left[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})R_{\beta i3} + X'_{33}M_{\beta 3} - \alpha'_{33}\alpha_{\beta 3}\right],\end{aligned}$$

$$\begin{aligned}\widetilde{m}^4_{\alpha} \coloneqq\ & m_{\alpha} - R_{\alpha i3}\Big[d'_{ij}\beta_{j3} - (\alpha'_{33}R_{3i3} + M'_{33}P_{3i3})p_3 + (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})m_3\Big] + \\ & -c'\alpha_{\alpha 3}\left[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})\beta_{i3} - M'_{33}p_3 + \alpha'_{33}m_3\right] + \\ & -c'M_{\alpha 3}\left[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})\beta_{i3} - X'_{33}m_3 + \alpha'_{33}p_3\right],\end{aligned}$$

$$\begin{aligned}\widetilde{c}^4_{v} \coloneqq\ & c_v + \beta_{i3}\Big[d'_{ij}\beta_{j3} + (\alpha'_{33}R_{3i3} - M'_{33}P_{3i3})p_3 + (\alpha'_{33}P_{3i3} - X'_{33}R_{3i3})m_3\Big] + \\ & -c'p_3\left[d_{ij}(M'_{33}P_{3j3} - \alpha'_{33}R_{3j3})\beta_{i3} - M'_{33}p_3 + \alpha'_{33}m_3\right] + \\ & -c'm_3\left[d_{ij}(X'_{33}R_{3j3} - \alpha'_{33}P_{3j3})\beta_{i3} - X'_{33}m_3 + \alpha'_{33}p_3\right].\end{aligned}$$

Appendix 2

Numerical Values of Material Coefficients

We record in Table 4.1 the numerical values of the material coefficients are presented for a transversely isotropic magneto-electro-thermo-elastic $BaTiO_3$–$CoFe_2O_4$ composite (see [9]).

Elastic moduli		**Magnetic permeabilities**	
$C_{1111} = C_{2222}$ (GPa)	200	$M_{11} = M_{22}$ (10^{-4} N s^2/C^2)	1.5
C_{1122} (GPa)	110	M_{33} (10^{-4} N s^2/C^2)	0.75
$C_{1133} = C_{2233}$ (GPa)	110	**Piezomagnetic constants**	
C_{3333} (GPa)	190	$R_{311} = R_{322}$ (N/A m)	200
$C_{2323} = C_{3131}$ (GPa)	45	R_{333} (N/A m)	260
C_{1212} (GPa)	45	R_{113} (N/A m)	180
Piezoelectric constants		**Magnetoelectric constants**	
$P_{311} = P_{322}$ (C/m^2)	-3.5	$\alpha_{11} = \alpha_{22}$ (10^{-12} N s/V C)	6
P_{333} (C/m^2)	11	α_{33} (10^{-12} N s/V C)	2500
Dielectric permittivities		**Pyroelectric constant**	
$X_{11} = X_{22}$ (10^{-9} C^2/N m^2)	0.9	p_3 (10^{-5} C/m^2 K)	-12.4
X_{33} (10^{-9} C^2/N m^2)	7.5	**Pyromagnetic constant**	
Thermal stresses		m_3 (10^{-3} N/A m K)	5.92
$\beta_{11} = \beta_{22}$ (10^6 N/K m^2)	4.86	**Density**	
β_{33} (10^6 N/K m^2)	4.32	ρ (kg/m^3)	5600
Thermal conductivity		**Calorific capacity**	
K_{33} (W/m K)	2.85	c_v (J/m^3 K^2)	325

Table 4.1 – Material properties of a thermo-piezo-electro-magnetic $BaTiO_3$–$CoFe_2O_4$ composite with 0.6 volume fraction of $BaTiO_3$.

Numerical Values of Reduced Material Coefficients

Table 4.2 presents the calculated numerical values of the reduced coefficients in the case of a plate behaving as an actuator, both piezoelectric and piezomagnetic. One can analogously obtain such values also in the other three cases.

Elastic moduli		**Magnetic permeability**	
$\widetilde{C}^3_{1111} = \widetilde{C}^2_{2222}$ (GPa)	136	$\widetilde{M}^3_{33}$ (10^{-4} N s^2/C^2)	0.75
$\widetilde{C}^3_{1122}$ (GPa)	46	**Piezomagnetic constants**	
$\widetilde{C}^3_{1212}$ (GPa)	45	$\widetilde{R}^3_{311} = \widetilde{R}^3_{322}$ (N/A m)	49.5
Piezoelectric constants		**Magnetoelectric constant**	
$\widetilde{P}^3_{311} = \widetilde{P}^3_{322}$ (C/m^2)	-9.86	$\widetilde{\alpha}^3_{33}$ (10^{-8} N s/V C)	1.75
Dielectric permittivity		**Pyroelectric constant**	
$\widetilde{X}^3_{33}$ (10^{-9} C^2/N m^2)	8.1	$\widetilde{p}^3_3$ (10^{-4} C/m^2 K)	2.5
Thermal stresses		**Pyromagnetic constant**	
$\widetilde{\beta}^3_{11} = \widetilde{\beta}^3_{22}$ (10^6 N/K m^2)	2.36	$\widetilde{m}^3_3$ (10^{-3} N/A m K)	5.91
		Calorific capacity	
		$\widetilde{c}^3_v$ (J/m^3 K^2)	423

Table 4.2 – Reduced coefficients for a magneto-electro-thermo-elastic actuator made up of a $BaTiO_3$–$CoFe_2O_4$ composite with 0.6 volume fraction of $BaTiO_3$.

Troisième partie

Simulation Numérique

Chapitre 5

Dynamique de Plaques en Flexion avec Inertie de Rotation

Nous choisissons maintenant de concentrer notre attention sur le problème de flexion qui caractérise tous les quatre problèmes de plaque déduits dans le chapitre précédent. L'intérêt de l'étude mathématique et, surtout, numérique de ce problème est lié au fait que l'équation d'évolution qui le régit tient compte d'un effet d'*inertie rotationnelle*. En effet, l'évolution est décrite par un terme de la forme $\frac{\partial^2}{\partial t^2}(\alpha w - \beta \Delta w)$, avec α et β des constantes réelles positives. Après avoir effectué un choix opportun des espaces fonctionnels, de telle sorte que le problème puisse être étudié en tant que problème hyperbolique abstrait, nous en montrons l'existence et l'unicité de la solution en suivant l'approche de Raviart et Thomas [54]. Ensuite nous effectuons l'analyse de l'erreur pour une discrétisation à éléments finis et pour une approximation en temps de type Newmark. Concernant la discrétisation en espace, nous utilisons une discrétisation *conforme* ; dans le cas des problèmes de plaque, il s'agit alors de considérer des éléments finis de classe C^1 ; nous choisissons d'utiliser les éléments HCT (voir e.g. [15]). Notre analyse numérique théorique est complétée et supportée avec des tests numériques effectués avec le logiciel `FreeFEM++`.

Ci-après, nous présentons la version étendue d'un article soumis à la revue *Journal of Numerical Mathematics*, écrit en collaboration avec G. Geymonat, F. Krasucki et M. Vidrascu. Remarquons que, pour simplifier les notations, dans la section suivante Ω représente le domaine bidimensionnel sur lequel le problème de flexion est formulé.

5.1 Mathematical and Numerical Modeling of Plate Dynamics with Rotational Inertia

General Notation

We denote by $\Omega \subset \mathbb{R}^2$ a two-dimensional domain with smooth boundary Γ; $\Gamma_0 \subset \Gamma$ is a measurable subset of Γ, with strictly positive length measure, and $\Gamma_1 = \Gamma \backslash \Gamma_0$. The outer unit normal and tangent vector fields on Γ are denoted, respectively, by $\mathbf{n} = (n_1, n_2)$ and $\boldsymbol{\tau} = (-n_2, n_1)$. For ψ a real-valued field defined on $\overline{\Omega}$, $\partial_n \psi = \nabla \psi \cdot \mathbf{n}$

and $\partial_\tau \psi = \nabla \psi \cdot \boldsymbol{\tau}$ denote its normal and tangential derivatives on Γ. The space dependence of a field is left tacit, unless noted otherwise. The time derivative of a real- or vector-valued field φ is denoted by $\dot{\varphi}$, of a function Φ taking values in a Hilbert space by $\frac{\mathrm{d}\Phi}{\mathrm{d}t}$.

In [10] the equations of plate models for magneto-electro-thermo-elastic sensors and actuators have been deduced by an asymptotic development with respect to the thickness $2\ell > 0$ of the plate. A peculiar feature of the different models concerns the flexural problem, governing the evolution of the transversal displacement w of the plate. It is *uncoupled* from the membrane problem, it takes into account an *inertia effect* involving the mean curvature of the deformed middle surface, and *the only influence* of magneto-electro-thermo-elastic behavior of the material appears in the coefficients $\mathbb{A} = (A_{\alpha\beta\sigma\tau})$ of the (symmetric) moment tensor $\mathbf{M}(t) = (M_{\alpha\beta}(t)) = -\mathbb{A}\nabla\nabla w(t)$, $(\nabla\nabla w(t))_{\alpha\beta} = \partial_{\alpha\beta} w(t)$ (let us point out that the fourth-order tensor $\mathbb{A} = (A_{\alpha\beta\sigma\tau})$ is symmetric and strongly elliptic). The vertical displacement w of the plate is solution of the following evolution equation:

$$2\ell\rho\ddot{w} - \frac{2\ell^3}{3}\rho\Delta\ddot{w} - \operatorname{div}\operatorname{div}\mathbf{M} = f + \operatorname{div}\mathbf{m} \quad \text{in } \Omega \times (0,T), \tag{5.1}$$

equipped with initial conditions

$$w(0) = w_0,\ \dot{w}(0) = w_1 \quad \text{in } \Omega \tag{5.2}$$

and with suitable boundary conditions. The influence of the rotational inertia on the lateral vibrations of linearly elastic bar was considered by Lord Rayleigh [55], sect. 186; the extension of this analysis to the flexural motion of isotropic elastic plates has been done for the first time in 1951 in a seminal paper by R. D. Mindlin [46] where also the effect of transverse shear deformation is taken into account. Later on, an evolution model for plates with rotational inertia was deduced by A. Raoult [52] using the asymptotic expansion method. The influence of these effects has also been considered in the case of flexural motion of large amplitude.

One can prove the existence and uniqueness of a weak solution (in a suitable Sobolev-valued distribution space) using a Faedo-Galerkin finite-dimensional approximation (see e.g. [3] and [21]) or else of a Sobolev-valued regular solution with a (semi)groups approach [56]. In both cases the presence of the rotational inertia term must be taken into account with an appropriate choice of the Sobolev space. Once this choice is done, the proof follows a well-known path. In sect. 5.1.1 we give the proof of the existence and uniqueness of the weak solution since this makes easier the subsequent numerical analysis. In sect. 5.1.2 we perform, in the appropriate Sobolev spaces, the error analysis of a finite element spatial discretization and of a Newmark-type discrete time approximation. Let us remark that our choice allows the application of the methods developed for linear second-order evolution equations. Based on a continuous-time Galerkin method (see e.g. [6] or [22]) we can infer optimal error estimates, and then couple with the error estimates for the time discretization of Newmark type (see e.g. [54]). Note that a conforming space discretization for plates requires C^1 elements; our choice are the HCT elements (see e.g. [15]). This theoretical numerical analysis is complemented with numerical tests performed with `FreeFEM++`.

5.1.1 Existence and Uniqueness

The weak formulation of the problem depends on the boundary conditions. For this, we first define the pivot space

$$H = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\} \tag{5.3}$$

and the bilinear form $b \colon H \times H \to \mathbb{R}$ given by

$$b(u, v) = 2\ell\rho \int_\Omega \left(uv + \frac{\ell^2}{3} \nabla u \cdot \nabla v \right) \mathrm{d}\Omega, \quad \forall u, v \in H. \tag{5.4}$$

Notice that $b(\cdot,\cdot)$ defines a scalar product in H whose associated norm $|\cdot|_b$ is equivalent to the usual Sobolev norm.

Let V be a Hilbert-Sobolev space such that

(i) $V \subseteq H^2(\Omega) \cap H$,
(ii) the embedding $V \hookrightarrow H$ is compact.

Thanks to (i), we can endow V with the Hessian L^2-norm given by

$$\|v\| = \left(\int_\Omega \nabla\nabla v : \nabla\nabla v \ \mathrm{d}\Omega \right)^{1/2}. \tag{5.5}$$

Let us define the bilinear form $a \colon V \times V \to \mathbb{R}$ by

$$a(u, v) = \int_\Omega \mathbb{A}\nabla\nabla u : \nabla\nabla v \ \mathrm{d}\Omega, \quad \forall u, v \in V. \tag{5.6}$$

By the symmetry and ellipticity properties of $\mathbb{A}$, $a(\cdot,\cdot)$ is symmetric and V-elliptic, i.e. there exists $\alpha > 0$ such that $a(v, v) \geqslant \alpha\|v\|^2$ for any $v \in V$. In the numerical examples, we consider essentially the following situations:

$$(BC)_1 : \begin{cases} \frac{2\ell^3}{3}\rho\, \partial_n \ddot{w} + \operatorname{div} \mathbf{M} \cdot \mathbf{n} + \partial_\tau(\mathbf{Mn} \cdot \boldsymbol{\tau}) = -\mathbf{m} \cdot \mathbf{n} & \text{on } \Gamma_1 \times (0, T), \\ \mathbf{Mn} \cdot \mathbf{n} = 0 & \text{on } \Gamma_1 \times (0, T), \\ w = \partial_n w = 0 & \text{on } \Gamma_0 \times (0, T), \end{cases}$$

$$(BC)_2 : \begin{cases} \frac{2\ell^3}{3}\rho\, \partial_n \ddot{w} + \operatorname{div} \mathbf{M} \cdot \mathbf{n} + \partial_\tau(\mathbf{Mn} \cdot \boldsymbol{\tau}) = -\mathbf{m} \cdot \mathbf{n} & \text{on } \Gamma_1 \times (0, T), \\ \mathbf{Mn} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T), \\ w = 0 & \text{on } \Gamma_0 \times (0, T). \end{cases}$$

Boundary conditions $(BC)_1$ refer to a plate *clamped* on Γ_0; in this case we choose

$$V = V_1 = \{u \in H^2(\Omega) : u = \partial_n u = 0 \text{ on } \Gamma_0\}. \tag{5.7}$$

Boundary conditions $(BC)_2$ feature a plate *simply supported* on Γ_0, in which case we choose

$$V = V_2 = \{u \in H^2(\Omega) : u = 0 \text{ on } \Gamma_0\} = H^2(\Omega) \cap H. \tag{5.8}$$

In both cases, under the general hypothesis of anisotropic linearly elastic behavior, the weak formulation of the problem has the following aspect:

$$\begin{cases} \text{For any fixed } t \in (0,T), \text{ find } w(t) \in V \text{ such that} \\ \displaystyle\int_\Omega \Big(2\ell\rho\, \ddot{w}(t) v + \frac{2\ell^3}{3} \rho\, \nabla \ddot{w}(t) \cdot \nabla v + \mathbb{A} \nabla\nabla w(t) : \nabla\nabla v \Big)\, \mathrm{d}\Omega \\ \qquad = \displaystyle\int_\Omega (f(t)v - \mathbf{m}(t) \cdot \nabla v)\, \mathrm{d}\Omega, \\ \text{for all } v \in V, \text{ with initial conditions } w(0) = w_0, \ \dot{w}(0) = w_1, \end{cases} \tag{5.9}$$

where $V = V_1$ or $V = V_2$. If the behavior is *isotropic*, the case to which we restrict our attention, the constitutive equation yielding $\mathbf{M}$ is given by a tensor $\mathbb{A}$ such that

$$\mathbb{A}\mathbf{T} = D\big((1-\nu)\mathbf{T} + \nu(\operatorname{tr}\mathbf{T})\,\mathbf{I}\big), \quad \forall \mathbf{T} \in \operatorname{Sym}(2), \tag{5.10}$$

where $\operatorname{Sym}(2)$ denotes the space of symmetric tensors on $\mathbb{R}^2$, and

$$D = \frac{2\ell^3 E}{3(1-\nu^2)}$$

is the *flexural rigidity* of the plate, with E and ν, respectively, the Young's modulus and the Poisson's ratio of the material. Equation (5.1) takes then the form

$$2\ell\rho\ddot{w} - \frac{2\ell^3}{3}\rho\,\Delta\ddot{w} + D\,\Delta\Delta w = f + \operatorname{div}\mathbf{m}, \tag{5.11}$$

and the weak formulation can be rewritten as

$$\begin{cases} \text{For any fixed } t \in (0,T), \text{ find } w(t) \in V \text{ such that} \\ \displaystyle\int_\Omega \Big(2\ell\rho\, \ddot{w}(t) v + \frac{2\ell^3}{3} \rho\, \nabla \ddot{w}(t) \cdot \nabla v + D(1-\nu)\, \nabla\nabla w(t) : \nabla\nabla v + \\ + D\nu\, \Delta w(t)\, \Delta v \Big) \mathrm{d}\Omega = \displaystyle\int_\Omega (f(t)v - \mathbf{m}(t) \cdot \nabla v)\, \mathrm{d}\Omega, \\ \text{for all } v \in V, \text{ with initial conditions } w(0) = w_0, \ \dot{w}(0) = w_1. \end{cases} \tag{5.12}$$

In order to show that the general problem (5.9) is well-posed, we identify the time-dependent linear form on H

$$L_t(v) = \int_\Omega (f(t)v - \mathbf{m}(t) \cdot \nabla v)\, \mathrm{d}\Omega,$$

with the scalar product of an element $F(t) \in H$ (for $0 < t < T$) with $v \in H$. This can be accomplished via the following problem:

$$\begin{aligned} &\text{Find } F(t) \in H \text{ such that} \\ &b(F(t), v) = L_t(v), \quad \forall v \in H, \end{aligned} \tag{5.13}$$

Provided $f(t) \in L^2(\Omega)$ and $\mathbf{m}(t) \in \mathbf{L}^2(\Omega)$ for $0 < t < T$, problem (5.13) is well-posed by the Lax-Milgram Lemma. As for problem (5.9), we assume $f \in L^2(0,T; L^2(\Omega))$

and $\mathbf{m} \in L^2(0,T;\mathbf{L}^2(\Omega))$, so that $F \in L^2(0,T;H)$; the formulation of the problem reads then:

$$\begin{aligned} \forall v \in V, \quad & \frac{\mathrm{d}^2}{\mathrm{d}t^2} b(w(t),v) + a(w(t),v) = b(F(t),v), \\ & w(0) = w_0, \ \frac{\mathrm{d}w}{\mathrm{d}t}(0) = w_1. \end{aligned} \tag{5.14}$$

Theorem 5.1. *Let $T > 0$ be fixed, $w_0 \in V$, $w_1 \in H$ and $F \in L^2(0,T;H)$.*

(i) *There exists a unique function $w \in C^0([0,T];V) \cap C^1([0,T];H)$ satisfying* (5.14).

(ii) *There exists a constant $c = c(T,\Omega)$ such that*
$\|w\|_{C^0([0,T];V)} + \|w\|_{C^1([0,T];H)} \leqslant c(T,\Omega)\Big\{\|w_0\| + |w_1|_b + \|F\|_{L^2(0,T;H)}\Big\}$.

Remark 5.1. The mappings $t \mapsto b(w(t),v)$ and $t \mapsto a(w(t),v)$ belong to $L^2(0,T)$, based on (i). Therefore, $(5.14)_1$ is to be understood *in the sense of distributions* on $(0,T)$.

Proof. (i) We proceed along the lines of Raviart and Thomas [54] (see also [3]).

Step 1. (*Uniqueness*) Owing to the compactness of the embedding $V \hookrightarrow H$, there exists an increasing sequence of eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \ldots$ and a Hilbert basis $\{g_i\}$, orthonormal in H and orthogonal [1] in V, of associated eigenvectors verifying

$$\forall v \in V, \quad a(g_i,v) = \lambda_i\, b(g_i,v) \tag{5.15}$$

(see, e.g., [56]). Let $\omega_i = \sqrt{\lambda_i}$ and $\Lambda\colon V \to H$ be the linear and continuous operator defined by

$$\forall v \in V, \quad \Lambda v = \sum_{i \geqslant 1} \omega_i\, b(v,g_i) g_i. \tag{5.16}$$

Then we have

$$|\Lambda v|_b = a(v,v)^{1/2}. \tag{5.17}$$

In fact, as $a(\cdot,\cdot)$ defines a scalar product in V, one has, for any $v \in V$,

$$a(v,v) = \sum_{i \geqslant 1} \omega_i^{-2} a(v,g_i)^2 = \sum_{i \geqslant 1} \omega_i^2\, b(v,g_i)^2 = \sum_{i \geqslant 1} b(\Lambda v,g_i)^2 = |\Lambda v|_b^2. \tag{5.18}$$

Moreover, for any $\theta \in \mathbb{R}$, let

$$\mathbf{Q}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

1. Recall that $|g_i|_b = 1$ and $\|g_i\|_a = \sqrt{\lambda_i}$, with $\|g_i\|_a = a(g_i,g_i)^{1/2}$, so that $\left\{\lambda_i^{-1/2} g_i\right\}$ is an orthonormal basis of V.

and for any $t \geqslant 0$, let the linear and continuous operator $\mathbb{G}(t)\colon H \times H \to H \times H$ be defined for any $\vec{v} = (v_1, v_2) \in H \times H$ by

$$\mathbb{G}(t)\vec{v} = \sum_{i\geqslant 1} \mathbf{Q}(\omega_i t) \begin{bmatrix} b(v_1, g_i) \\ b(v_2, g_i) \end{bmatrix} g_i = \tag{5.19}$$

$$= \sum_{i\geqslant 1} \begin{pmatrix} \{b(v_1, g_i)\cos(\omega_i t) + b(v_2, g_i)\sin(\omega_i t)\} g_i \\ \{-b(v_1, g_i)\sin(\omega_i t) + b(v_2, g_i)\cos(\omega_i t)\} g_i \end{pmatrix}. \tag{5.20}$$

If w is solution of (5.14), then it verifies

$$\begin{pmatrix} \Lambda w(t) \\ \dfrac{\mathrm{d}w}{\mathrm{d}t}(t) \end{pmatrix} = \mathbb{G}(t) \begin{pmatrix} \Lambda w_0 \\ w_1 \end{pmatrix} + \int_0^t \mathbb{G}(t-s) \begin{pmatrix} 0 \\ F(s) \end{pmatrix} \mathrm{d}s. \tag{5.21}$$

Indeed, as $w \in C^0([0,T]; V)$, we have

$$w(t) = \sum_{i\geqslant 1} b(w(t), g_i) g_i, \quad a(w(t), g_i) = \lambda_i\, b(w(t), g_i)$$

so that $\alpha_i(t) = b(w(t), g_i)$ is the unique solution to the following Cauchy problem:

$$\begin{cases} \ddot{\alpha}_i(t) + \lambda_i \alpha_i(t) = b(F(t), g_i), \\ \alpha_i(0) = b(w_0, g_i), \quad \dot{\alpha}_i(0) = b(w_1, g_i), \end{cases} \tag{5.22}$$

whose explicit form is given by

$$\alpha_i(t) = b(w_0, g_i)\cos(\omega_i t) + \frac{1}{\omega_i} b(w_1, g_i)\sin(\omega_i t) + \frac{1}{\omega_i}\int_0^t \sin(\omega_i(t-s))\, b(F(s), g_i)\,\mathrm{d}s.$$

Hence α_i verifies

$$\begin{bmatrix} \omega_i \alpha_i(t) \\ \dot{\alpha}_i(t) \end{bmatrix} = \mathbf{Q}(\omega_i t) \begin{bmatrix} \omega_i\, b(w_0, g_i) \\ b(w_1, g_i) \end{bmatrix} + \int_0^t \mathbf{Q}(\omega_i(t-s)) \begin{bmatrix} 0 \\ b(F(s), g_i) \end{bmatrix} \mathrm{d}s, \tag{5.23}$$

whose rows, by (5.16) and (5.19), are the i-th components of the corresponding rows of (5.21). Therefore w is given by the series development

$$w(t) = \sum_{i\geqslant 1} \Big\{ b(w_0, g_i)\cos(\omega_i t) + \frac{1}{\omega_i} b(w_1, g_i)\sin(\omega_i t) + \\ + \frac{1}{\omega_i}\int_0^t \sin\left(\omega_i(t-s)\right)\, b(F(s), g_i)\,\mathrm{d}s \Big\} g_i,$$

whence we infer that, if the solution to (5.14) exists, it is unique.

Step 2. (*Existence*) Let us introduce the subspace V_m of V generated by the first m eigenvectors $g_1, \ldots, g_m$. We seek a function $w_m\colon t \in [0,T] \mapsto w_m(t) \in V_m$ solution to the following second-order system of differential equations:

$$\forall v_m \in V_m, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} b(w_m(t), v) + a(w_m(t), v) = b(F(t), v), \tag{5.24}$$

$$w_m(0) = w_{0,m} = \sum_{i=1}^{m} b(w_0, g_i) g_i, \quad \frac{\mathrm{d}w_m}{\mathrm{d}t}(0) = w_{1,m} = \sum_{i=1}^{m} b(w_1, g_i) g_i. \tag{5.25}$$

On writing

$$w_m(t) = \sum_{i=1}^{m} \alpha_i(t) g_i, \quad \alpha_i(t) = b(w_m(t), g_i),$$

one can repeat the same arguments as in Step 1 to see that α_i is solution of (5.22) and hence verifies (5.23).

Now, $\{w_m\}$ is a Cauchy sequence in spaces $C^0([0,T];V)$ and $C^1([0,T];H)$ (and thus it converges in such spaces as they are complete). To see this, let $m, p \in \mathbb{N}$ with $p > m \geqslant 1$; from (5.18) we have

$$\begin{aligned} a(w_p(t) - w_m(t), w_p(t) - w_m(t)) + \left| \frac{\mathrm{d}}{\mathrm{d}t} \left(w_p(t) - w_m(t) \right) \right|_b^2 \\ = \sum_{i=m+1}^{p} \left(\lambda_i \alpha_i(t)^2 + \dot{\alpha}_i(t)^2 \right). \end{aligned}$$

Since $\mathbf{Q}(\omega_i t)$ is an orthogonal matrix, we infer from (5.23) that

$$\left(\lambda_i \alpha_i(t)^2 + \dot{\alpha}_i(t)^2 \right)^{1/2} \leqslant \left(\lambda_i\, b(w_0, g_i)^2 + b(w_1, g_i)^2 \right)^{1/2} + \int_0^t |b(F(s), g_i)| \, \mathrm{d}s,$$

whence, by Jensen's inequality,

$$\lambda_i \alpha_i(t)^2 + \dot{\alpha}_i(t)^2 \leqslant 2 \left\{ \lambda_i\, b(w_0, g_i)^2 + b(w_1, g_i)^2 + t \int_0^t b(F(s), g_i)^2 \, \mathrm{d}s \right\};$$

finally, on taking the sum on i ranging from $m+1$ to p,

$$\begin{aligned} a(w_p(t) - w_m(t), w_p(t) - w_m(t)) + \left| \frac{\mathrm{d}}{\mathrm{d}t} \left(w_p(t) - w_m(t) \right) \right|_b^2 \\ \leqslant 2 \sum_{i=m+1}^{p} \left\{ \lambda_i\, b(w_0, g_i)^2 + b(w_1, g_i)^2 + t \int_0^t b(F(s), g_i)^2 \, \mathrm{d}s \right\}. \end{aligned}$$

As $w_0 \in V$, $w_1 \in H$ and $F \in L^2(0,T;H)$, the series of the general terms appearing on the right-hand side are convergent, and hence

$$\lim_{m,p \to +\infty} \sum_{i=m+1}^{p} \left\{ \lambda_i\, b(w_0, g_i)^2 + b(w_1, g_i)^2 + T \int_0^T b(F(s), g_i)^2 \, \mathrm{d}s \right\} = 0.$$

Finally, as $a(\cdot,\cdot)$ is V-elliptic, $\{w_m\}$ is a Cauchy sequence in $C^0([0,T];V)$ and $C^1([0,T];H)$. Thus, there exists a function w such that

$$w_m \to w \text{ in } C^0([0,T];V) \cap C^1([0,T];H) \quad \text{as } m \to +\infty. \tag{5.26}$$

It remains to prove that w is solution to (5.14). To see this, take $\psi \in \mathcal{D}(0,T)$ and select an integer $\mu \geqslant 1$. From (5.24), for any $m \geqslant \mu$, we have

$$\forall v \in V_\mu, \ \int_0^T b(w_m(t), v) \frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}(t) \, \mathrm{d}t + \int_0^T a(w_m(t), v) \psi(t) \, \mathrm{d}t = \int_0^T b(F(t), v) \psi(t) \, \mathrm{d}t,$$

and passing to the limit as $m \to +\infty$, thanks to (5.26), we get

$$\forall v \in V_\mu, \quad \int_0^T b(w(t), v)\frac{d^2\psi}{dt^2}(t)\,dt + \int_0^T a(w(t), v)\psi(t)\,dt = \int_0^T b(F(t), v)\psi(t)\,dt. \tag{5.27}$$

As $\bigcup_{\mu\geqslant 1} V_\mu$ is dense in V, (5.27) holds for any $v \in V$ and we get $(5.14)_2$. From (5.26) we also have

$$w_m(0) \to w(0) \text{ in } V, \quad \frac{dw_m}{dt}(0) \to \frac{dw}{dt}(0) \text{ in } H.$$

Finally, by (5.25),

$$w_m(0) = w_{0,m} \to w_0 \text{ in } V, \quad \frac{dw_m}{dt}(0) = w_{1,m} \to w_1 \text{ in } H,$$

whence we get $(5.14)_3$ as well.

(ii) From the orthogonality of $\mathbf{Q}(\omega_i t)$ and the definition (5.19), we infer that for any $\vec{v} = (v_1, v_2) \in H \times H$,

$$\|\mathbb{G}(t)\vec{v}\|_{H\times H} = \left\{\sum_{i\geqslant 1}\left((v_1, g_i)^2 + (v_2, g_i)^2\right)\right\}^{1/2} = \|\vec{v}\|_{H\times H}.$$

Then, from (5.21) one has that

$$\left\{|\Lambda w(t)|_b^2 + \left|\frac{du}{dt}(t)\right|_b^2\right\}^{1/2} \leqslant \left\{|\Lambda w_0|_b^2 + |w_1|_b^2\right\}^{1/2} + \int_0^t |F(s)|_b\,ds,$$

and the result follows from (5.17) and from the V-ellipticity of $a(\cdot,\cdot)$. □

5.1.2 Numerical Analysis

5.1.2.1 Semi-Discrete Problem

Let $V_h \subset V$ denote a subspace of V of dimension $I = I(h)$ and consider the following semi-discrete problem: *given $w_{0,h} \in V_h$ and $w_{1,h} \in V_h$, find the solution $w_h : t \in [0,T] \mapsto w_h(t) \in V_h$ to the following system of ordinary differential equations*:

$$\begin{aligned} \forall v_h \in V_h, \quad & \frac{d^2}{dt^2} b(w_h(t), v_h) + a(w_h(t), v_h) = b(F(t), v_h), \\ & w_h(0) = w_{0,h}, \ \frac{dw_h}{dt}(0) = w_{1,h}. \end{aligned} \tag{5.28}$$

We now introduce a basis $\{\varphi_i\}_{1\leqslant i\leqslant I}$ of V_h and denote the time-dependent components of w_h in this basis by $\xi_j(t)$, $j = 1, \ldots, I$. Analogously, we denote the components in the same basis of $w_{0,h}$ and $w_{1,h}$ respectively by $\xi_{0,j}$ and $\xi_{1,j}$. Finally, we set $\chi_j(t) = b(F(t), \varphi_j)$. Then (5.28) reads

$$\begin{cases} \mathcal{M}_h\ddot{\boldsymbol{\xi}}(t) + \mathcal{K}_h\boldsymbol{\xi}(t) = \boldsymbol{\chi}(t), \\ \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \quad \dot{\boldsymbol{\xi}}(0) = \boldsymbol{\xi}_1, \end{cases} \tag{5.29}$$

with self-explanatory notation. Matrices $\mathcal{M}_h$ and $\mathcal{K}_h$ are respectively the *mass matrix* and the *stiffness matrix*. Their coefficients are

$$(\mathcal{M}_h)_{ij} = b(\varphi_i, \varphi_j) \quad \text{and} \quad (\mathcal{K}_h)_{ij} = a(\varphi_i, \varphi_j), \quad 1 \leqslant i, j \leqslant I. \tag{5.30}$$

In the following, we drop the index h when there is no ambiguity.

Remark 5.2. The choice of the scalar product (5.4) in H ensures that $\mathcal{M}$ is positive definite.

Problem (5.29) is the *semi-discrete* counterpart of (5.14). Under certain in-time regularity hypotheses on w and adapting the arguments of [6], [22], it is possible to give an estimate of the error $w_h(t) - w(t)$. For this, we introduce the *elliptic projection operator* Π_h, a linear and continuous operator mapping $u \in V$ onto $\Pi_h u \in V_h$ defined by

$$\forall v_h \in V_h, \quad a(\Pi_h u - u, v_h) = 0.$$

We have the following results.

Theorem 5.2. *Let $T > 0$ be fixed and assume that the solution w of* (5.14) *verifies $w \in C^2([0,T]; V)$. Then there exists a constant $C = C(T)$ independent of h such that, for any $t \in [0,T]$,*

$$\|w_h(t) - w(t)\| + \left|\frac{\mathrm{d}w_h}{\mathrm{d}t}(t) - \frac{\mathrm{d}w}{\mathrm{d}t}(t)\right|_b \leqslant C\Bigg\{ \|w_{0,h} - \Pi_h w_0\| + |w_{1,h} - \Pi_h w_1|_b +$$

$$+ \|(\mathrm{I} - \Pi_h) w(t)\| + \left|(\mathrm{I} - \Pi_h)\frac{\mathrm{d}w}{\mathrm{d}t}(t)\right|_b + \int_0^t \left|(\mathrm{I} - \Pi_h)\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}(s)\right|_b \mathrm{d}s \Bigg\}.$$

Theorem 5.3. *Under the assumptions of Theorem 5.2 let the following* approximation hypotheses *be satisfied:*

$$\forall v \in V, \quad \lim_{h \to 0} \inf_{v_h \in V_h} \|v - v_h\| = 0, \tag{5.31}$$

$$\lim_{h \to 0} \|w_{0,h} - w_0\| = 0, \tag{5.32}$$

$$\lim_{h \to 0} |w_{1,h} - w_1|_b = 0; \tag{5.33}$$

then

$$\forall t \in [0,T], \quad \lim_{h \to 0} \left\{ \|w_h(t) - w(t)\| + \left|\frac{\mathrm{d}w_h}{\mathrm{d}t}(t) - \frac{\mathrm{d}w}{\mathrm{d}t}(t)\right|_b \right\} = 0.$$

This convergence result can actually be improved; under the hypotheses of Theorem (5.1),

$$\lim_{h \to 0} w_h = w \quad \text{in } C^0([0,T]; V) \text{ and in } C^1([0,T]; H).$$

5.1.2.2 Time discretization: Newmark's method

A typical time discretization method widely used for differential systems of the form (5.29) is the *Newmark method* (see, e.g., [26]), of which we recall the main features.

Let $N \in \mathbb{N}$; we introduce a time-step $\Delta t = \frac{T}{N}$ and a uniform mesh of the interval $[0,T]$ defined by nodes $t_n = n\Delta t,\ 0 \leqslant n \leqslant N$. We want to compute an approximation

$\left(\boldsymbol{\xi}^n, \dot{\boldsymbol{\xi}}^n\right)$ of the pair $\left(\boldsymbol{\xi}(t_n), \dot{\boldsymbol{\xi}}(t_n)\right)$. The Newmark method consists in making the following approximation assumptions [26, 54]:

$$\begin{cases} \dot{\boldsymbol{\xi}}^{n+1} = \dot{\boldsymbol{\xi}}^n + \Delta t \left((1-\gamma)\ddot{\boldsymbol{\xi}}^n + \gamma\ddot{\boldsymbol{\xi}}^{n+1}\right), \\ \boldsymbol{\xi}^{n+1} = \boldsymbol{\xi}^n + \Delta t\, \dot{\boldsymbol{\xi}}^n + \Delta t^2 \left(\left(\dfrac{1}{2} - \beta\right)\ddot{\boldsymbol{\xi}}^n + \beta\ddot{\boldsymbol{\xi}}^{n+1}\right), \end{cases} \qquad 0 \leqslant n \leqslant N-1,$$

where we have denoted by $\ddot{\boldsymbol{\xi}}^n$ an approximation of $\ddot{\boldsymbol{\xi}}(t_n)$ and γ, β are parameters. These relationships are *implicit*, as the acceleration $\ddot{\boldsymbol{\xi}}^{n+1}$ needs to be determined in order to find $\boldsymbol{\xi}^{n+1}$ and $\dot{\boldsymbol{\xi}}^{n+1}$. Typical choices for γ and β are

$$(\gamma, \beta) = \left(\frac{1}{2}, \frac{1}{4}\right), \quad (\gamma, \beta) = \left(\frac{1}{2}, \frac{1}{6}\right);$$

the first choice corresponds to the assumption of *constant* acceleration within the time interval $[t_n, t_{n+1})$, the second one presumes acceleration to be *linear* over the same interval.

Applied to (5.29), the Newmark method reads

$$\frac{1}{\Delta t^2}\boldsymbol{\mathcal{M}}(\boldsymbol{\xi}^{n+2} - 2\boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^n) + \boldsymbol{\mathcal{K}}\left(\beta\boldsymbol{\xi}^{n+2} + \left(\frac{1}{2} - 2\beta + \gamma\right)\boldsymbol{\xi}^{n+1} + \left(\frac{1}{2} + \beta - \gamma\right)\boldsymbol{\xi}^n\right) =$$
$$= \beta\boldsymbol{\chi}(t_{n+2}) + \left(\frac{1}{2} - 2\beta + \gamma\right)\boldsymbol{\chi}(t_{n+1}) + \left(\frac{1}{2} + \beta - \gamma\right)\boldsymbol{\chi}(t_n), \quad 0 \leqslant n \leqslant N-2,$$

$$\frac{1}{\Delta t^2}\boldsymbol{\mathcal{M}}\left(\boldsymbol{\xi}^1 - \boldsymbol{\xi}_0 - \Delta t\,\boldsymbol{\xi}_1\right) + \boldsymbol{\mathcal{K}}\left(\beta\boldsymbol{\xi}^1 + \left(\frac{1}{2} - \beta\right)\boldsymbol{\xi}_0\right) = \beta\boldsymbol{\chi}(t_1) + \left(\frac{1}{2} - \beta\right)\boldsymbol{\chi}(t_0),$$

which is equivalent, on denoting by $w_h^n \in V_h$ an approximation of $w_h(t_n) \in V$, to the following system:

$$\forall v_h \in V_h, \quad \frac{1}{\Delta t^2}\, b\left(w_h^{n+2} - 2w_h^{n+1} + w_h^n, v_h\right) +$$
$$+\, a\left(\beta w_h^{n+2} + \left(\frac{1}{2} - 2\beta + \gamma\right) w_h^{n+1} + \left(\frac{1}{2} + \beta - \gamma\right) w_h^n, v_h\right) =$$
$$= b\left(\beta F(t_{n+2}) + \left(\frac{1}{2} - 2\beta + \gamma\right) F(t_{n+1}) +\right.$$
$$\left. + \left(\frac{1}{2} + \beta - \gamma\right) F(t_n), v_h\right), \quad 0 \leqslant n \leqslant N-2,$$

$$\forall v_h \in V_h, \quad \frac{1}{\Delta t^2}\, b\left(w_h^1 - w_{0,h} - \Delta t\, w_{1,h}, v_h\right) + a\left(\beta w_h^1 + \left(\frac{1}{2} - \beta\right) w_{0,h}, v_h\right) =$$
$$= b\left(\beta F(t_1) + \left(\frac{1}{2} - \beta\right) F(t_0), v_h\right). \tag{5.34}$$

Hence, at every time step, a linear system of the form

$$\left(\boldsymbol{\mathcal{M}} + \beta\Delta t^2\boldsymbol{\mathcal{K}}\right)\boldsymbol{\xi}^{n+1} = \boldsymbol{\eta}^n,$$

where $\boldsymbol{\eta}^n \in \mathbb{R}^I$ is known, has to be solved; provided $\beta \geqslant 0$, $\boldsymbol{\mathcal{M}} + \beta\Delta t^2\boldsymbol{\mathcal{K}}$ is a symmetric and positive definite matrix.

Concerning the stability of the method applied to (5.29), as well as error estimates, the following result holds.

Theorem 5.4. *Let $\beta \geqslant 0$, $\delta = \gamma - \frac{1}{2} \geqslant 0$, L a constant independent of h and Δt, and $0 < \varepsilon < 1$. Then the solution $\{w_h^n \in V_h, 0 \leqslant n \leqslant N\}$ of* (5.34) *verifies the following estimates:*

(i) *provided $w \in C^2([0,T];V) \cap C^3([0,T];H)$ and the* stability condition

$$\Delta t^2 \, \lambda_{I,h} \leqslant \begin{cases} L & \text{if } \beta \geqslant \dfrac{(1+\delta)^2}{4}, \\ \dfrac{4}{(1+\delta^2)-4\beta}(1-\varepsilon) & \text{if } \beta < \dfrac{(1+\delta)^2}{4}, \end{cases} \tag{5.35}$$

it holds

$$\begin{aligned} |w_h^n - w(t_n)|_b \leqslant C\Big\{ & |w_{0,h} - \Pi_h w_0|_b + |w_{1,h} - \Pi_h w_1|_b + |(\mathrm{I} - \Pi_h) w(t_n)|_b + \\ & + \int_0^{t_n} \left(\left| (\mathrm{I}-\Pi_h) \frac{\mathrm{d}^2 w}{\mathrm{d}t^2}(s) \right|_b + \Delta t \left| \frac{\mathrm{d}^3 w}{\mathrm{d}t^3}(s) \right|_b \right) \mathrm{d}s \Big\}, \end{aligned} \tag{5.36}$$

where the constant $C = C(T)$ is independent of h, Δt and w (it depends on L and ε);

(ii) *if $\delta = 0$, $w \in C^2([0,T];V) \cap C^4([0,T];H)$ and* (5.35) *holds, we have*

$$\begin{aligned} |w_h^n - w(t_n)|_b \leqslant C\Big\{ & |w_{0,h} - \Pi_h w_0|_b + |w_{1,h} - \Pi_h w_1|_b + |(\mathrm{I} - \Pi_h) w(t_n)|_b + \\ & + \int_0^{t_n} \left(\left| (\mathrm{I}-\Pi_h) \frac{\mathrm{d}^2 w}{\mathrm{d}t^2}(s) \right|_b + \Delta t^2 \left| \frac{\mathrm{d}^4 w}{\mathrm{d}t^4}(s) \right|_b \right) \mathrm{d}s \Big\}, \end{aligned} \tag{5.37}$$

where the constant $C = C(T)$ is independent of h, Δt and w.

When $\beta \geqslant (1+\delta)^2/4$, the stability condition [2] $\Delta t^2 \, \lambda_{I,h} \leqslant L$ does not impose real restrictions on the choice of the time-step, inasmuch as constant L is actually arbitrary. Thus, provided $\gamma \geqslant \frac{1}{2}$, for $\beta \geqslant \frac{1}{4}\left(\frac{1}{2}+\gamma\right)^2$ the Newmark method is *unconditionally stable*. Furthermore, for $(\gamma, \beta) = \left(\frac{1}{2}, \frac{1}{4}\right)$, condition $(5.35)_1$ is even unnecessary. We make this choice of the parameters in the following, in which case the discretization of (5.29) can be rewritten as

$$\begin{cases} \left(\boldsymbol{\mathcal{M}} + \dfrac{\Delta t^2}{4} \boldsymbol{\mathcal{K}} \right) \boldsymbol{\xi}^{n+\frac{1}{2}} = \boldsymbol{\mathcal{M}} \boldsymbol{\xi}^n + \dfrac{\Delta t}{2} \boldsymbol{\mathcal{M}} \dot{\boldsymbol{\xi}}^n + \dfrac{\Delta t^2}{4} \boldsymbol{\chi}^{n+\frac{1}{2}}, \ 0 \leqslant n \leqslant N-1, \\ \boldsymbol{\xi}^0 = \boldsymbol{\xi}_0, \quad \dot{\boldsymbol{\xi}}^0 = \boldsymbol{\xi}_1 \end{cases} \tag{5.38}$$

2. We recall that $\lambda_{I,h}$ is the maximum eigenvalue satisfying the discrete analogous of (5.15).

with $\boldsymbol{\xi}^{n+\frac{1}{2}} = \frac{1}{2}\left(\boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^{n}\right)$ and $\boldsymbol{\chi}^{n+\frac{1}{2}} = \frac{1}{2}\left(\boldsymbol{\chi}(t_{n+1}) + \boldsymbol{\chi}(t_n)\right)$. Upon solving (5.38) for $\boldsymbol{\xi}^{n+\frac{1}{2}}$, the update rules are

$$\dot{\boldsymbol{\xi}}^{n+1} = \frac{4}{\Delta t}\left(\boldsymbol{\xi}^{n+\frac{1}{2}} - \boldsymbol{\xi}^{n}\right) - \dot{\boldsymbol{\xi}}^{n},$$
$$\boldsymbol{\xi}^{n+1} = 2\boldsymbol{\xi}^{n+\frac{1}{2}} - \boldsymbol{\xi}^{n}.$$

This scheme is called *Newmark's midpoint approximation*. In this situation, Raviart and Thomas [54] have proved the following error estimate.

Theorem 5.5. *Let $T > 0$ be fixed. Then the solution $\{w_h^n \in V_h, 0 \leqslant n \leqslant N\}$ given by* (5.38) *verifies the following estimates:*

(i) *if $w \in C^2([0,T];V) \cap C^3([0,T];H)$,*

$$|w_h^n - w(t_n)|_b \leqslant C\{|w_{0,h} - \Pi_h w_0|_b + |w_{1,h} - \Pi_h w_1|_b + $$
$$+|(I-\Pi_h)w(t_n)|_b + \int_0^{t_n}\left\{\left|(I-\Pi_h)\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}(s)\right|_b + \Delta t\left|\frac{\mathrm{d}^3 w}{\mathrm{d}t^3}(s)\right|_b\right\}\mathrm{d}s\},$$

where $C = C(T)$ is independent of h, Δt and w;

(ii) *if $w \in C^2([0,T];V) \cap C^4([0,T];H)$,*

$$|w_h^n - w(t_n)|_b \leqslant C\{|w_{0,h} - \Pi_h w_0|_b + |w_{1,h} - \Pi_h w_1|_b + $$
$$+|(I-\Pi_h)w(t_n)|_b + \int_0^{t_n}\left\{\left|(I-\Pi_h)\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}(s)\right|_b + \Delta t^2\left|\frac{\mathrm{d}^4 w}{\mathrm{d}t^4}(s)\right|_b\right\}\mathrm{d}s\},$$

where $C = C(T)$ is independent of h, Δt and w.

5.1.2.3 Space discretization: HCT Elements

The solution of (5.14) belongs to $C^0([0,T];V) \cap C^1([0,T];H)$, thus we are led to select *finite elements of class C^1*; in particular, we will use HCT elements.

Let $\mathcal{T}_h$ denote a regular mesh in the sense of Ciarlet [15, 54] of the domain $\overline{\Omega}$, $K \in \mathcal{T}_h$ the typical element of $\mathcal{T}_h$ and X_h a finite element space. Moreover, let $P_K = \{v_{h|K} : v_h \in X_h\}$. We recall then [15] the following result.

Theorem 5.6. *Assume that the inclusions $P_K \subset H^2(K)$ for all $K \in \mathcal{T}_h$ and $X_h \subset C^1(\overline{\Omega})$ hold. Then the following inclusions hold:*

$$X_h \subset H^2(\Omega),$$
$$X_{oh} = \{v_h \in X_h : v_h = 0 \text{ on } \Gamma_0\} \subset V_2,$$
$$X_{ooh} = \{v_h \in X_h : v_h = \partial_n v_h = 0 \text{ on } \Gamma_0\} \subset V_1.$$

Remark 5.3. The choices $V_h = X_{ooh}$ for $(BC)_1$ and $V_h = X_{oh}$ for $(BC)_2$ ensure hypotheses (5.31) to (5.33) of Theorem 5.3 to be satisfied [15].

Finite elements of class C^1 are rather complicated and time-consuming, and are not used too often in practical applications. We choose to start our experiments with such elements because the theory described is valid for conforming approximations.

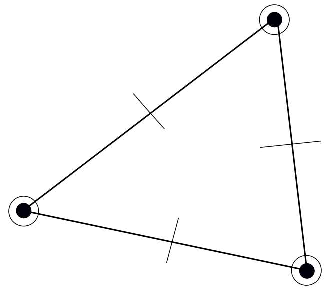


Figure 5.1 – The HCT element.

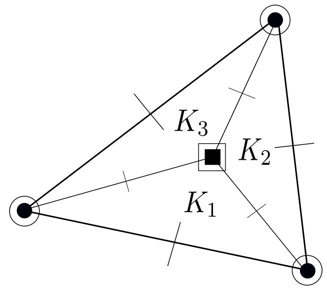


Figure 5.2 – The HCT element in detail.

In this context, the HCT is one of the simplest C^1 elements (Fig. 5.1). The set of degrees of freedom (twelve in total) is given by the values of a function, as well as of its partial derivatives, at the three vertices and by the values of its normal derivatives at the midpoints of the sides.

From an internal viewpoint, the HCT element is a composite element: a typical triangle $K \in \mathcal{T}_h$ is split into three sub-triangles K_i ($i = 1, 2, 3$), the internal node usually corresponding to the barycenter of K (Fig. 5.2). A polynomial of degree three is defined on each sub-triangle, so that the space P_K is given by

$$P_K = \{p \in C^1(K) : p_{|K_i} \in \mathbb{P}_3(K_i),\ 1 \leqslant i \leqslant 3\}.$$

The condition $p \in C^1(K)$ is realized by requiring the continuity of the three polynomial expansions and of their gradients at the barycenter (marked by a black square in Fig. 5.2), and the continuity of their normal derivatives at the midpoints of the internal sides. Thus, the HCT element is a C^1 element as a whole, in the sense that a function and its first derivatives are continuous across the edges of any two adjacent elements of $\mathcal{T}_h$.

Let us now provisionally focus on the static counterpart of (5.14), namely,

$$\forall v \in V, \quad a(w, v) = L(v), \tag{5.39}$$

where we have neglected time dependence. Notice that in this case the pivot space H is $L^2(\Omega)$. The discrete version of (5.39) reads

$$\sum_{j=1}^{I} a(\varphi_i, \varphi_j)\xi_j = L(\varphi_i), \quad 1 \leqslant i \leqslant I.$$

The following theorem [15] yields an estimate of the error between w_h and w when HCT elements are employed in space discretization for (5.39).

Theorem 5.7. *If the exact solution $w \in V$ of* (5.39) *is also in the space $H^4(\Omega)$, then there exists a constant $C > 0$ independent of h such that*

$$\|w - w_h\|_{H^2(\Omega)} \leqslant Ch^2 |w|_{H^4(\Omega)}. \tag{5.40}$$

Remark 5.4 (Implementation issues). In finite element methods, a *quadrature scheme* is needed to compute the coefficients $a(\varphi_i, \varphi_j)$ and $L(\varphi_i)$, thereby resulting in an *approximated bilinear form $a_h(\cdot,\cdot)$* and in an *approximated linear form $L_h(\cdot)$*. Generally, integration over a mesh element is performed using a quadrature scheme for which all nodes are situated at the interior of the element. However, in the HCT case, a mesh element features internal interfaces between any two sub-triangles; at these interfaces, the continuity of second partial derivatives is not guaranteed. Hence, one should use a quadrature scheme that avoids nodes on any such interface. A solution is to *integrate on each sub-triangle and then sum up the three contributions*. Moreover, the following *Theorem (first Strang lemma)* requires that the integration error and the interpolation error be of the same order, and hence the quadrature scheme must be chosen accordingly.

Theorem 5.8. *Let the bilinear form $a_h(\cdot,\cdot)$ be* uniformly V_h-elliptic*, i.e.*

$$\exists \alpha > 0 : \ \forall v_h \in V_h, \quad a_h(v_h, v_h) \geqslant \alpha \|v_h\|^2,$$

where α is independent of V_h. Then there exists a constant $C > 0$ independent of V_h such that

$$\|w - w_h\| \leqslant C \left(\inf_{v_h \in V_h} \left\{ \|w - v_h\| + \sup_{u_h \in V_h} \frac{|a(v_h, u_h) - a_h(v_h, u_h)|}{\|u_h\|} \right\} + \right.$$
$$\left. + \sup_{u_h \in V_h} \frac{|L(u_h) - L_h(u_h)|}{\|u_h\|} \right).$$

If the space P_K contains polynomials of degree at most k, a sufficient condition for $a_h(\cdot,\cdot)$ to be uniformly V_h-elliptic is that *the quadrature scheme used be exact for polynomials of degree $2k - 4$ at least* [15]. Thus, in our case, we have to use on each $K_i \subset K$ a quadrature formula exact at least for polynomials of degree two.

5.1.3 Numerical Results

We perform numerical tests using the software package `FreeFEM++ 3.42` (see [37]), in which HCT elements have been implemented along with an adequate quadrature formula for their use. We will consider two situations:

(1) $\Omega = \Omega^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\}$, $\Gamma_0 = \partial\Omega$ and $(BC)_1$,
i.e. a circular plate of radius R clamped all over the lateral surface;

(2) $\Omega = \Omega^2 = (0, a) \times (0, b)$ with $a, b > 0$, $\Gamma_0 = \partial\Omega$ and $(BC)_2$,
i.e. a rectangular plate simply supported all over the lateral surface.

In both situations we assume $\mathbf{m} = \mathbf{0}$, and in order to get coherent results we fix once and for all the following set of data:

$$R = 5\,cm, \ a = 6\,cm, \ b = 8\,cm, \ \ell = 1\,mm$$
$$\rho = 5600 \ kg/m^3, \ E = 136\,GPa, \ \nu = 0.3, \tag{5.41}$$
$$\text{(i.e. } D = 99.63\,N\cdot m).$$

$\|w - w_h\|_{H^2(\Omega)}$	$4.22 \cdot 10^{-7}$	$1.20 \cdot 10^{-7}$	$3.40 \cdot 10^{-8}$	$9.53 \cdot 10^{-9}$
Number of finite elements	115	460	1840	7360
Number of degrees of freedom	398	1483	5723	22483

Table 5.1 – Variation of the H^2 norm of the error with the number of finite elements for w given by (5.43), along with the corresponding number of degrees of freedom. Computations have been made for $f_0 = -130\, N/m^2$ (this value corresponds to the weight of the plate per unit area of the middle surface).

5.1.3.1 Statics

In each of the following test cases, we consider *nested meshes* in order to determine the behavior of $\|w - w_h\|_{H^2(\Omega)}$ with respect to the meshsize h; of course, mesh refinement is uniform.

Boundary conditions $(BC)_1$

The problem formulation reads in this case

$$\begin{cases} D\,\Delta\Delta w = f & \text{in } \Omega, \\ w = \partial_n w = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.42}$$

where $\Omega = \Omega^1$. We first consider a test case for which f is constant over Ω ($f \equiv f_0$); the closed-form solution of (5.42) is given in this case by

$$w(x, y) = \frac{f_0}{64D}(R^2 - (x^2 + y^2))^2. \tag{5.43}$$

As Table 5.1 shows, the implementation of HCT elements in `FreeFEM++ 3.42` is quite effective, as the variation of the H^2 norm (regardless of physical dimensions) of the error with respect to the meshsize is in agreement with error estimate (5.40).

As a second test case, in order to deal with a non-polynomial solution, we consider

$$w(x, y) = \frac{f_0}{64D}(R^2 - (x^2 + y^2))^2 \sin(ax), \tag{5.44}$$

a being a constant, which is the solution corresponding to

$$f(x, y) = \frac{f_0}{64}\left\{-16ax(-8 + a^2(-R^2 + x^2 + y^2))\cos(ax) + \right.$$
$$\left. + \, (64 + 16a^2(2R^2 - 5x^2 - 3y^2) + a^4(-R^2 + x^2 + y^2)^2)\sin(ax)\right\}.$$

The variation of the error is shown in Table 5.2, and the convergence is again quadratic.

$\|w - w_h\|_{H^2(\Omega)}$	$2.00 \cdot 10^{-6}$	$6.01 \cdot 10^{-7}$	$1.70 \cdot 10^{-7}$	$4.61 \cdot 10^{-8}$
Number of finite elements	115	460	1840	7360
Number of degrees of freedom	398	1483	5723	22483

Table 5.2 – Variation of the H^2 norm of the error with the number of finite elements for w given by (5.44), along with the corresponding number of degrees of freedom. Computations have been made for $f_0 = -130\, N/m^2$ and $a = 0.5\, cm^{-1}$.

Boundary conditions $(BC)_2$

The problem formulation is

$$\begin{cases} D\,\Delta\Delta w = f & \text{in } \Omega, \\ w = 0,\ \mathbf{Mn}\cdot\mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.45}$$

where $\Omega = \Omega^2$. In this case, when the source term is of the form

$$f(x, y) = f_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right),$$

the closed-form solution is given by

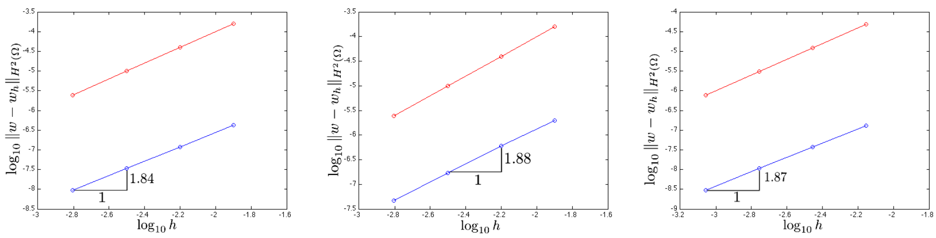
$$w(x, y) = W_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right), \quad W_0 = \frac{f_0}{\pi^4 D}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{-2}. \tag{5.46}$$

The variation of the error is shown in Table 5.3, and once more the convergence is quadratic.

$\|w - w_h\|_{H^2(\Omega)}$	$1.30\cdot 10^{-7}$	$3.78\cdot 10^{-8}$	$1.08\cdot 10^{-8}$	$2.96\cdot 10^{-9}$
Number of finite elements	214	856	3424	13696
Number of degrees of freedom	725	2731	10595	41731

Table 5.3 – Variation of the H^2 norm of the error with the number of finite elements for w given by (5.46), along with the corresponding number of degrees of freedom. Computations have been made for $f_0 = -130\,N/m^2$.

We have represented in Fig. 5.3 the behavior of the error in the three test cases we illustrated.



(a) (b) (c)

Figure 5.3 – Error H^2-norm vs. meshsize in a logarithmic scale for w as in (5.43) (a), (5.44) (b) and (5.46) (c). The red line represents theoretical behavior, and has slope 2.

5.1.3.2 Dynamics

In order to test the accuracy of the Newmark method combined with HCT elements, we consider the time evolution of the vertical displacement $t \mapsto w_h(x_0, y_0; t)$, where (x_0, y_0) is the center of the plate, as the mesh is uniformly refined – again, we consider nested meshes – in the two cases $(BC)_1$ and $(BC)_2$. The influence of the time-step Δt is also pointed out in some test cases for which it turns out to be relevant. The data set is given in (5.41). When the exact solution $(x, y; t) \mapsto w(x, y; t)$ is known, we consider the evolution of the error

$$e_h(t) = |w_h(x_0, y_0; t) - w(x_0, y_0; t)|. \tag{5.47}$$

When the source term is nonzero, we can consider an exact solution $w\colon \overline{\Omega} \times [0, T] \to \mathbb{R}$ of the form

$$w(x, y; t) = k(x, y)g(t),$$

where $g\colon [0, T] \to \mathbb{R}$ is a C^2-class function and $k\colon \overline{\Omega} \to \mathbb{R}$ is solution of the corresponding static problem. In particular, we test the robustness of the numerical method in the following three cases:

(a) $g(t) = \alpha t^2$,

(b) $g(t) = \alpha \arctan(\beta t)$,

(c) $g(t) = \alpha \sin(\beta t)$,

where α and β are constants. Since convergence occurs only for $0 < t \leqslant T$ where T is fixed, we precise in each example the value of T.

Boundary conditions $(BC)_1$

Nonzero source term

The solution (5.43) of (5.42) corresponding to a constant surface load can be used to derive a closed-form solution of the dynamic problem with a nonzero and time-dependent source term. A straightforward computation shows that

$$w(x, y; t) = \frac{f_0}{64D}(R^2 - (x^2 + y^2))^2 g(t), \tag{5.48}$$

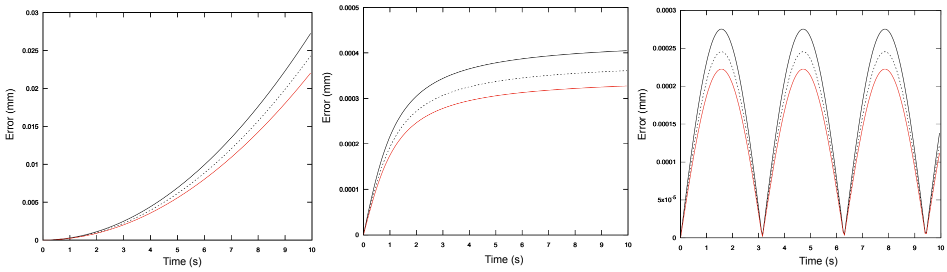
with f_0 a constant, is solution to the dynamic problem when

$$f(x, y; t) = \frac{f_0 \ell \rho}{96D}\Big(3(R^2 - (x^2 + y^2))^2 + 8\ell^2(R^2 - 2(x^2 + y^2))\Big)\ddot{g}(t) + f_0\, g(t).$$

The evolution of the error given by (5.47) corresponding to each of the three choices of g is shown in Fig. 5.4, and it reflects the expected behavior: the error evolution is attenuated upon refining the mesh.

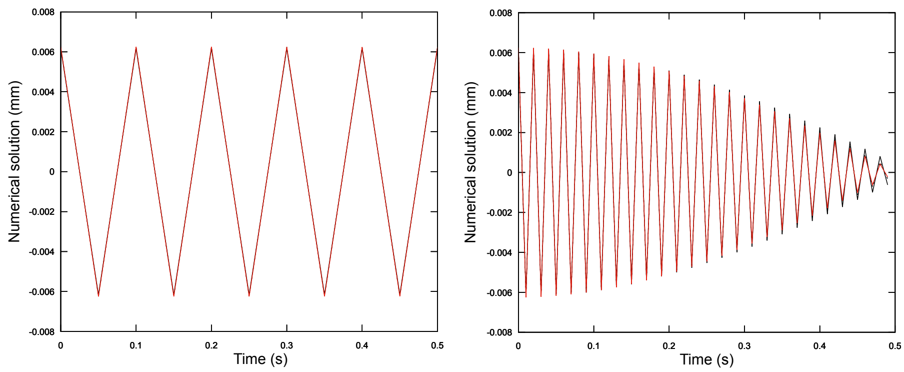
Vanishing source term

When $f \equiv 0$, we cannot use (5.48). We have thus compared in Fig. 5.5 the evolution of the numerical solutions in $(0, 0)$ corresponding to three nested meshes, and for two choices of the time-step ($\Delta t = 0.05\, s$ and $\Delta t = 0.01\, s$), in the case of nonzero initial displacement and vanishing initial velocity: $w_0(x, y) = \alpha(R^2 - (x^2 + y^2))^2$ and $w_1 \equiv 0$, with α a constant. An analogous comparison is made in Fig. 5.6, in the case of vanishing initial displacement and nonzero initial velocity: $w_0 \equiv 0$ and $w_1(x, y) = \alpha(R^2 - (x^2 + y^2))$, with α a constant. Notice that, in any case, there is a complete overlapping between the three solutions; on the other hand, the effect of the time-step is very remarkable: on the same time interval, for $\Delta t = 0.01\, s$ one captures many more oscillations than for $\Delta t = 0.05\, s$.



(a) (b) (c)

Figure 5.4 – Evolution of the error (in mm) $t \mapsto e_h(t)$ with $(x_0, y_0) = (0, 0)$, for $g(t) = \alpha t^2$ (a), $g(t) = \alpha \arctan \beta t$ (b), $g(t) = \alpha \sin \beta t$ (c), for $\alpha = 10$ and $\beta = 1\, s^{-1}$. Continuous line: 13 elements; dashed line: 52 elements; red line: 208 elements. In all cases, $f_0 = -130\, N/m^2$ and $T = 10\, s$. Time-step: $\Delta t = 0.05\, s$.



(a) (b)

Figure 5.5 – Comparison between three time evolutions (in mm) $t \mapsto w_h(0, 0; t)$ for a nonzero initial displacement and a vanishing initial velocity, for a time-step $\Delta t = 0.05\, s$ (a) and a time-step $\Delta t = 0.01\, s$ (b). Continuous line: 13 elements; dashed line: 52 elements; red line: 208 elements. In both cases, $\alpha = 10^{-6}\, cm^{-3}$ and $T = 0.5\, s$.

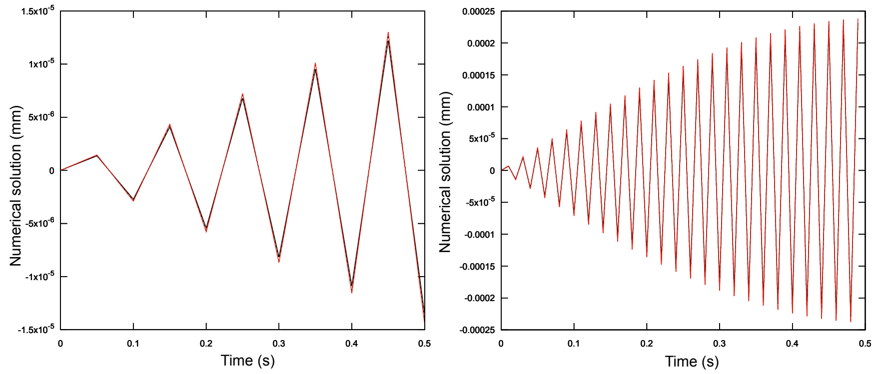


Figure 5.6 – Comparison between three time evolutions (in mm) $t \mapsto w_h(0,0;t)$ for a nonzero initial velocity and a vanishing initial displacement, for a time-step $\Delta t = 0.05\,s$ (a) and a time-step $\Delta t = 0.01\,s$ (b). Continuous line: 13 elements; dashed line: 52 elements; red line: 208 elements. In both cases $\alpha = 10^{-2}\,cm^{-1}s^{-1}$ and $T = 0.5\,s$.

Boundary conditions $(BC)_2$

Nonzero source term

In order to consider a situation with $f \neq 0$, one can take, analogously to (5.48),

$$w(x, y; t) = W_0 \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{b}y\right) g(t) \tag{5.49}$$

with W_0 as in (5.46), so that w is solution to the dynamic problem with

$$f(x, y; t) = \frac{2a^2b^2 f_0 \ell \left(b^2\ell^2\pi^2 + a^2(3b^2 + \ell^2\pi^2)\right)}{3\pi^4 D(a^2 + b^2)^2} \rho \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{b}y\right) \ddot{g}(t) + \\ + f_0 \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{b}y\right) g(t).$$

The evolution of the error given by (5.47) corresponding to each of the three choices of g is shown in Fig. 5.7. Let us remark that in all of the three cases, the behavior of the error corresponding to the finest mesh (represented by a red line) is almost imperceptible, inasmuch as it is very close to zero.

Vanishing source term

When $f \equiv 0$ the solution can be obtained by separation of variables. Indeed, one obtains the Fourier development

$$w(x, y; t) = \sum_{m,n \in \mathbb{N}} \left(g^1_{mn} \cos(\omega_{mn} t) + g^2_{mn} \sin(\omega_{mn} t)\right) \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right),$$

where

$$\omega_{mn} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \sqrt{\frac{D}{2\ell\rho + \frac{2}{3}\pi^2\ell^3\rho\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}},$$

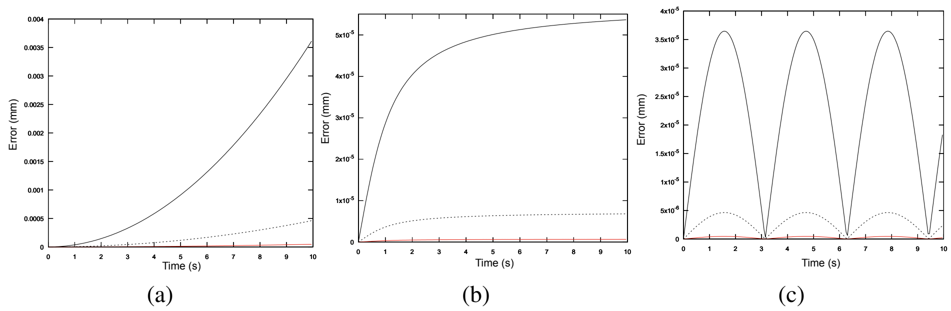


Figure 5.7 – Evolution of the error (in mm) $t \mapsto e_h(t)$, with $(x_0, y_0) = (a/2, b/2)$, for $g(t) = \alpha t^2$ (a), $g(t) = \alpha \arctan \beta t$ (b), $g(t) = \alpha \sin \beta t$ (c), for $\alpha = 10$ and $\beta = 1\,s^{-1}$. Continuous line: 12 elements; dashed line: 48 elements; red line: 192 elements. In all cases, $f_0 = -130\,N/m^2$ and $T = 10\,s$. Time-step: $\Delta t = 0.05\,s$.

and coefficients g^1_{mn} and g^2_{mn} are determined by initial conditions. We consider then the following two situations:

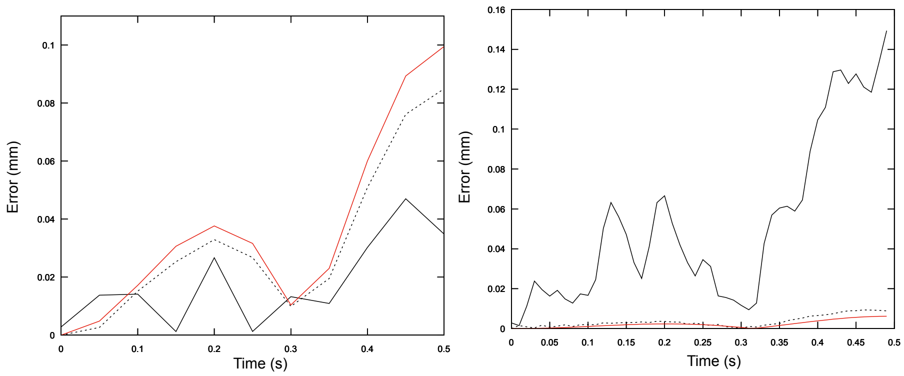
— $w_0(x, y) = \alpha \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{b}y\right)$ and $w_1(x, y) \equiv 0$, in which case the exact solution is
$$w(x, y; t) = \alpha \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{b}y\right) \cos(\omega_{11}t);$$

— $w_0(x, y) \equiv 0$ and $w_1(x, y) = \alpha \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{b}y\right)$, in which case the exact solution is
$$w(x, y; t) = \frac{\alpha}{\omega_{11}} \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{b}y\right) \sin(\omega_{11}t).$$

Given that the exact solution in both cases is $(2\pi/\omega_{11})$-periodic in time, in order to test our numerical method we consider a reasonable value of ω_{11}; say, $\omega_{11} = 10\,s^{-1}$. The evolution of the error given by (5.47) corresponding to the first case is shown in Fig. 5.8 for a time-step $\Delta t = 0.05\,s$ and a time-step $\Delta t = 0.01\,s$; an analogous test has been made for the second case, represented in Fig. 5.9. For $\Delta t = 0.05\,s$, the obtained behavior is unexpected: mesh refinement results in an amplification of the error evolution; decreasing the time-step to $\Delta t = 0.01\,s$ yields the expected behavior. Indeed, note that the period τ corresponding to $\omega_{11} = 10\,s^{-1}$ is approximately equal to $0.63\,s$, so that the ratio $\Delta t/\tau$ is about 0.08 for $\Delta t = 0.05\,s$ and about 0.01 for $\Delta t = 0.01\,s$.

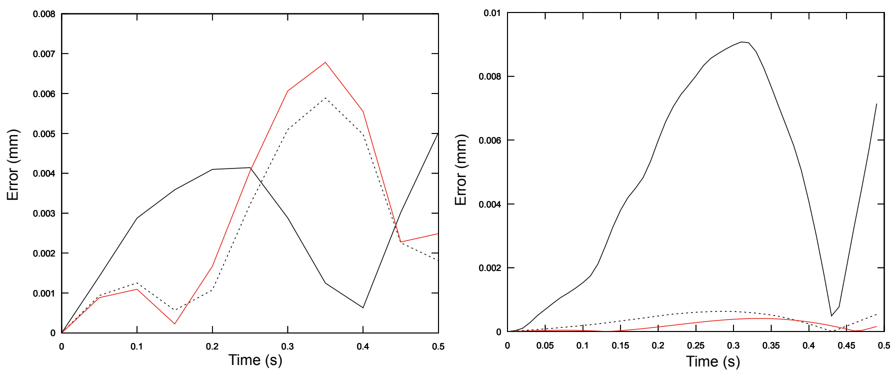
Concluding Remarks

Our numerical tests performed using `FreeFEM++ 3.42` show that the presence of the rotational inertia term does not affect the efficiency of the Newmark time discretization method combined with conforming finite elements such as HCT elements. As is well-known, HCT elements are computationally expensive; it would then be of interest to use nonconforming space discretization methods (mixed or hybrid) [8, 51], such as HHO methods [18, 19]. The study of the use of such methods for plate dynamics problems in presence of rotational inertia seems particularly interesting and will be carried out in a forthcoming work.



(a) (b)

Figure 5.8 – Evolution of the error (in mm) $t \mapsto e_h(t)$, with $(x_0, y_0) = (a/2, b/2)$, between the analytical and numerical solutions for a nonzero initial displacement and a vanishing initial velocity, for a time-step $\Delta t = 0.05\,s$ (a) and a time-step $\Delta t = 0.01\,s$ (b). Continuous line: 13 elements; dashed line: 52 elements; red line: 208 elements. In both cases, $\alpha = 1\,mm$ and $T = 0.5\,s$.



(a) (b)

Figure 5.9 – Evolution of the error (in mm) $t \mapsto e_h(t)$, with $(x_0, y_0) = (a/2, b/2)$, between the analytical and numerical solutions for a vanishing initial displacement and a nonzero initial velocity, for a time-step $\Delta t = 0.05\,s$ (a) and a time-step $\Delta t = 0.01\,s$ (b). Continuous line: 13 elements; dashed line: 52 elements; red line: 208 elements. In both cases, $\alpha = 1\,mm/s$ and $T = 0.5\,s$.

Quatrième partie

Développements et Perspectives

Chapitre 6

Introduction aux Méthodes de Discrétisation HHO pour le Problème de Flexion

On présente dans ce chapitre quelques préliminaires à l'utilisation d'une méthode de discrétisation non conforme en espace pour le traitement numérique du problème de plaque en flexion rencontré dans les deux chapitres précédents. Plus précisement, il s'agit d'une méthode *hybride et d'ordre élevé* (HHO, Hybrid High-Order). Les méthodes hybrides d'ordre élevé sont formulées en termes d'inconnues discrètes définies sur les faces et sur les cellules du maillage (d'où le terme "hybride"), et de telles inconnues sont des polynômes de degré arbitraire $k \geqslant 0$ (d'où le terme "élevé").

La présentation suivante a été largement inspirée par l'ouvrage de Di Pietro et Ern [17], par les deux articles de Di Pietro, Ern et Lemaire [18, 19], et par l'article de Di Pietro et Droniou [20]. Nous considérons dans toute la suite le problème (bidimensionnel) de plaque en flexion suivant :

$$\begin{cases} -\operatorname{div}\operatorname{div}\mathbf{M} = f & \text{dans } \Omega, \\ u = \partial_n u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{6.1}$$

où $\mathbf{M}$ est le tenseur de moment, lié à l'inconnue u (le déplacement vertical) par la loi de comportement $\mathbf{M} = -\mathbb{A}\nabla\nabla u$, $\mathbb{A}$ étant un champ tensoriel du quatrième ordre symétrique et uniformément elliptique. Pour simplifier, on considérera toujours $\mathbb{A}$ constant sur Ω.

Rappelons, avec référence au chapitre précédent, la formulation variationnelle de (6.1) :

$$\text{Étant donné } f \in L^2(\Omega), \text{ trouver } u \in H^2_0(\Omega) \text{ tel que}$$
$$\int_\Omega \mathbb{A}\nabla\nabla u : \nabla\nabla v \,\mathrm{d}\Omega =: a(u,v) = l(v) := \int_\Omega f v \,\mathrm{d}\Omega, \quad \forall v \in H^2_0(\Omega). \tag{6.2}$$

La méthode de discrétisation dans ce chapitre étant non conforme, l'espace de dimension finie auquel la solution approchée u_h appartient *n'est pas inclus dans* $H^2_0(\Omega)$; le deuxième gradient $\nabla\nabla u_h$ ne serait donc pas défini. C'est pourquoi la méthode est basée sur un opérateur de reconstruction qui reproduit les propriétés du deuxième gradient au niveau discret. On considère dans ce qui suit des maillages réguliers, dont on rappelle les propriétés dans la section suivante.

6.1 Maillages Réguliers

Soit $\mathcal{H} \subset \mathbb{R}_+^*$ un ensemble dénombrable de pas de maillage ayant 0 comme seul point d'accumulation. Dans la suite, on simplifie la présentation en considérant, pour chaque $h \in \mathcal{H}$, une triangulation $\mathcal{T}_h$ du domaine $\overline{\Omega}$ satisfaisant les conditions de régularité usuelles de Ciarlet [15]. Plus précisément, $\mathcal{T}_h$ est une collection finie de triangles telle que $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$ et $h := \max_{T\in\mathcal{T}_h} h_T$, avec h_T le diamètre de T. Une face F est définie comme un segment fermé de $\overline{\Omega}$ de mesure positive et tel que (i) soit il existe $T_1, T_2 \in \mathcal{T}_h$ tels que $F \subset \partial T_1 \cap \partial T_2$, auquel cas F est une interface, soit (ii) il existe $T \in \mathcal{T}_h$ tel que $F \subset \partial T \cap \partial\Omega$, et F est une face de bord. On note $\mathcal{F}_h^i$ l'ensemble des interfaces, $\mathcal{F}_h^b$ l'ensemble des faces de bord, et on pose $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^b$. Le diamètre d'une face $F \in \mathcal{F}_h$ est noté h_F. Pour tout $T \in \mathcal{T}_h$, $\mathcal{F}_T := \{F \in \mathcal{F}_h : F \subset \partial T\}$ est l'ensemble des faces du bord de l'élément T et, pour tout $F \in \mathcal{F}_T$, $\mathbf{n}_{TF}$ est la normale unitaire à F sortante de T. Symétriquement, pour tout $F \in \mathcal{F}_h$, on note $\mathcal{T}_F := \{T \in \mathcal{T}_h : F \subset \partial T\}$ l'ensemble des éléments qui partagent la face F (deux si F est une interface, un si F est une face de bord).

Remarque 6.1. On peut considérer une situation plus générale, dans laquelle chaque élément $T \in \mathcal{T}_h$ est un polygone constitué par la réunion d'un nombre fini de triangles réguliers, au sens définit par Ciarlet [15].

Soit $\varrho > 0$ le paramètre de régularité du maillage, c'est à dire, pour tout $h \in \mathcal{H}$ [17],

$$\varrho^2 h_T \leqslant h_F \leqslant h_T.$$

On montre [17] qu'il existe un entier N_∂ dépendant de ϱ tel que

$$\forall h \in \mathcal{H}, \quad \max_{T\in\mathcal{T}_h} \operatorname{card}(\mathcal{F}_T) \leqslant N_\partial.$$

Il existe aussi des nombres réels C_{tr} et $C_{tr,c}$ dépendants de ϱ mais indépendants de h tels que pour tout $T \in \mathcal{T}_h$ et $F \in \mathcal{F}_T$, l'on ait les deux inégalités suivantes (respectivement, inégalité de trace discrète et inégalité de trace continue)

$$\begin{aligned} \|v\|_F &\leqslant C_{tr}\, h_F^{-1/2}\, \|v\|_T & \forall v \in \mathbb{P}_d^k(T), \\ \|v\|_{\partial T} &\leqslant C_{tr,c} \left(h_T^{-1}\|v\|_T^2 + h_T\|\nabla v\|_T^2\right)^{1/2} & \forall v \in H^1(T), \end{aligned} \tag{6.3}$$

où l'on a noté $\|\cdot\|_X$ la norme dans $L^2(X)$ avec $X \subset \overline{\Omega}$ (on notera dans la suite $(\cdot,\cdot)_X$ le produit scalaire dans $L^2(X)$ ou dans $\mathbf{L}^2(X)$), et $\mathbb{P}_d^\ell(X)$ l'espace des restrictions à X des polynômes de degré $\leqslant \ell$ en d variables ($d = 1, 2$).

Un élément essentiel dans la construction de la méthode de discrétisation sont les projecteurs orthogonaux L^2 sur des espaces de polynômes locaux définis sur des sous-ensembles bornés $U \subset \mathbb{R}^2$. On introduit alors le projecteur L^2 comme l'opérateur $\pi_X^\ell : L^2(U) \to \mathbb{P}_d^\ell(U)$ défini par

$$\int_U \pi_U^\ell v\, w = \int_U v w, \quad \forall w \in \mathbb{P}_d^\ell(X).$$

Rappelons un résultat de [18] : il existe un nombre réel $C(\varrho, \ell) > 0$ indépendant de h tel que, pour tout $T \in \mathcal{T}_h$, tout $s \in \{1, \ldots, \ell+1\}$ et tout $v \in H^s(T)$, l'on ait

$$|v - \pi_T^\ell v|_{H^m(T)} + h_T^{1/2} |v - \pi_T^\ell v|_{H^m(\partial T)} \leqslant C(\varrho, \ell) h_T^{s-m} |v|_{H^s(T)}, \ \forall m \in \{0, \ldots, s-1\}, \tag{6.4}$$

où $|\cdot|_{H^m(T)}$ et $|\cdot|_{H^m(\partial T)}$ sont les semi-normes dans les espaces respectifs. Dans la suite, on abrégera comme $a \lesssim b$ toute inégalité de la forme $a \leqslant Cb$ avec $C > 0$ indépendant de h mais éventuellement dépendant d'autres paramètres.

6.2 Aspects Locaux : Degrés de Liberté, Interpolation et Reconstruction

Soit $k \geqslant 2$ un entier fixé.

(i) L'*espace local des degrés de liberté* (DDL) est défini comme l'ensemble

$$\underline{U}_T^k := \mathbb{P}_2^k(T) \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_1^{k+1}(F) \right\} \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_1^k(F) \right\}. \tag{6.5}$$

Dans la suite on n'écrit pas les indices 1 et 2 afin de simplifier les notations. Pour une collection générale de degrés de liberté $\underline{v}_T \in \underline{U}_T^k$, on utilise la notation soulignée

$$\underline{v}_T = (v_T, (v_{n,F})_{F \in \mathcal{F}_T}, (v_F)_{F \in \mathcal{F}_T}),$$

où v_T est le DDL lié à l'inconnue de maille, $v_{n,F}$ celui lié à la dérivée normale de l'inconnue sur la face F, et v_F celui lié à l'inconnue sur la face F.

(ii) L'*opérateur d'interpolation locale* $\underline{I}_T^k : H^2(T) \to \underline{U}_T^k$ est défini par

$$\underline{I}_T^k v = \left(\pi_T^k v, (\pi_F^{k+1} \partial_{n_{TF}} v)_{F \in \mathcal{F}_T}, (\pi_F^k v)_{F \in \mathcal{F}_T} \right), \quad \forall v \in H^2(T), \tag{6.6}$$

où $\partial_{n_{TF}} v$ désigne la dérivée normale de v sur la face F.

Remarque 6.2. Puisque le bord de T est régulier par morceaux, le théorème de trace assure que v est bien définie sur chaque face F, ainsi que sa dérivée normale $\partial_{n_{TF}} v$.

Pour $u, v \in H^2(T)$, posons $a_T(u, v) := (\mathbb{A} \nabla\nabla u, \nabla\nabla v)_T$.

(iii) L'*opérateur de reconstruction locale* $p_{\Delta,T}^k : \underline{U}_T^k \to \mathbb{P}^{k+2}(T)$ est défini par la solution du problème variationnel local suivant :

Trouver $p_{\Delta,T}^k \underline{v}_T \in \mathbb{P}^{k+2}(T)$ tel que, pour tout $w \in \mathbb{P}^{k+2}(T)$,

$$\begin{aligned} a_T(p_{\Delta,T}^k \underline{v}_T, w) = \quad & -(v_T, \operatorname{div}\operatorname{div} \mathbf{M}_w)_T - \sum_{F \in \mathcal{F}_T} (v_{n,F}, \mathbf{M}_w \mathbf{n}_{TF} \cdot \mathbf{n}_{TF})_F + \\ & + \sum_{F \in \mathcal{F}_T} \left(v_F, \operatorname{div} \mathbf{M}_w \cdot \mathbf{n}_{TF} + \partial_\tau (\mathbf{M}_w \mathbf{n}_{TF} \cdot \boldsymbol{\tau}) \right)_F, \end{aligned} \tag{6.7}$$

où $\mathbf{M}_w := -\mathbb{A} \nabla\nabla w$, $\boldsymbol{\tau}$ est le vecteur unitaire tangent sur ∂T et ∂_τ la dérivée tangentielle sur ∂T.

Remarque 6.3. On utilise ici la notation $\mathbf{M}_w$ pour remarquer le fait que $\mathbf{M}_w$ est un tenseur de moment de nature *virtuelle* (l'espace des déplacements virtuels étant constitué de polynômes de degré $k+2$ sur T), à la différence du tenseur $\mathbf{M}$ qui apparaît dans la forme $a_T(\cdot,\cdot)$, introduit au début du chapitre.

Le problème (6.7) admet une solution dans $\mathbb{P}^{k+2}(T)$, comme $a_T(\cdot,\cdot)$ est une forme bilinéaire continue, symétrique et positive sur cet espace. Notons, de plus, que l'opérateur $\nabla\nabla\colon \mathbb{P}^{k+2}(T) \to [\mathbb{P}^k(T)]^4$ a pour noyau l'ensemble des polynômes de degré 1 sur T :

$$\ker \nabla\nabla = \mathbb{P}^1(T).$$

Une condition nécessaire et suffisante pour l'existence de la solution est alors que la forme linéaire au second membre de (6.7) s'annule sur les éléments de $\ker\nabla\nabla$; puisque

$$\mathbf{M}_w = \mathbf{0} \quad \forall w \in \mathbb{P}^1(T),$$

la condition est satisfaite. La solution de (6.7) n'est donc pas unique : si $p^k_{\Delta,T}\underline{v}_T \in \mathbb{P}^{k+2}(T)$ est une solution, $p^k_{\Delta,T}\underline{v}_T + d$ pour tout $d \in \mathbb{P}^1(T)$ l'est aussi.

Le problème (6.7) est un problème linéaire en dimension *finie*, égale à $\dim \mathbb{P}^{k+2}(T) = (k+4)(k+3)/2$. On peut expliciter le système d'équations linéaires correspondant (que l'on peut assimiler à un problème de *pseudo-flexion* local). Pour cela, on utilise les formules d'intégration par parties sur les termes du premier et du second membre, et on obtient le système linéaire suivant :

$$\begin{cases} -\mathrm{div}\,\mathrm{div}\,\mathbf{M}^k_{\Delta,T} = -\mathrm{div}\,\mathrm{div}\,\mathbf{M}_{v_T} & \text{dans } T,\\ \mathbf{M}^k_{\Delta,T}\mathbf{n}_{TF}\cdot\mathbf{n}_{TF} = \mathbf{M}_{v_T}\mathbf{n}_{TF}\cdot\mathbf{n}_{TF} + \mathfrak{L}^n v_{n,F} & \text{sur chaque } F \in \mathcal{F}_T,\\ \mathrm{div}\,\mathbf{M}^k_{\Delta,T}\cdot\mathbf{n}_{TF} + \partial_\tau(\mathbf{M}^k_{\Delta,T}\mathbf{n}_{TF}\cdot\boldsymbol{\tau}) = & \\ = \mathrm{div}\,\mathbf{M}_{v_T}\cdot\mathbf{n}_T + \partial_\tau(\mathbf{M}^k_{v_T}\mathbf{n}_T\cdot\boldsymbol{\tau}) + \mathfrak{L}v_F & \text{sur chaque } F \in \mathcal{F}_T,\end{cases}$$

où $\mathbf{M}^k_{\Delta,T} \coloneqq -\mathbb{A}\nabla\nabla p^k_{\Delta,T}\underline{v}_T$, $\mathbf{M}_{v_T} \coloneqq -\mathbb{A}\nabla\nabla v_T$, $(\mathfrak{L}^n v_{n,F}, w)_F \coloneqq (v_{n,F}, \mathbf{M}_w\mathbf{n}_{TF}, \mathbf{n}_{TF})_F$ et $(\mathfrak{L}v_F, w)_F \coloneqq \big(v_F, \mathrm{div}\,\mathbf{M}_w\cdot\mathbf{n}_{TF} + \partial_\tau(\mathbf{M}_w\mathbf{n}_{TF}\cdot\boldsymbol{\tau})\big)_F$.

Pour déterminer $p^k_{\Delta,T}\underline{v}_T$ de façon unique, on rajoute la *condition de fermeture* suivante :

$$\pi^1_T p^k_{\Delta,T}\underline{v}_T = \pi^1_T v_T. \tag{6.8}$$

Le condition (6.8) s'interprète comme suit. On écrit la solution de (6.7) comme $p^k_{\Delta,T}\underline{v}_T = \tilde{u}+a+bx+cy$, avec $\tilde{u} \in \mathbb{P}^{k+2}(T)$ tel que $(\tilde{u}, w)_T = 0$ pour tout $w \in \mathbb{P}^1(T)$, et $a, b, c \in \mathbb{R}$. Puisqu'une base de $\mathbb{P}^1(T)$ est donnée par $\{1, x, y\}$, la condition de fermeture (6.8) est équivalente au système linéaire suivant :

$$\begin{pmatrix} \mathrm{mes}(T) & \displaystyle\int_T x & \displaystyle\int_T y \\ \displaystyle\int_T x & \displaystyle\int_T x^2 & \displaystyle\int_T xy \\ \displaystyle\int_T y & \displaystyle\int_T xy & \displaystyle\int_T y^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \displaystyle\int_T v_T \\ \displaystyle\int_T xv_T \\ \displaystyle\int_T yv_T \end{pmatrix}.$$

Ainsi, les trois constantes a, b, c sont uniquement déterminées en fonction des moments polynômiaux d'ordre inférieur ou égal à 1 du DDL de maille v_T.

6.3 Consistance Polynômiale Locale

Puisque $\mathbb{P}^{k+2}(T) \subset H^2(T)$, $p^k_{\Delta,T}\underline{I}^k_T$ est un opérateur linéaire et continu dans $H^2(T)$. Le lemme suivant donne une propriété remarquable de cet opérateur.

Lemme 6.1. *Pour tout $v \in H^2(T)$, on a*

$$a_T(v - p^k_{\Delta,T}\underline{I}^k_T v, w) = 0, \quad \forall w \in \mathbb{P}^{k+2}(T). \tag{6.9}$$

Démonstration. On écrit (6.7) pour $\underline{v}_T = \underline{I}^k_T v = (\pi^k_T v, (\pi^{k+1}_F \partial_{n_{TF}} v)_{F\in\mathcal{F}_T}, (\pi^k_F v)_{F\in\mathcal{F}_T})$. Puisque $w \in \mathbb{P}^{k+2}(T)$ et que $\mathbb{A}$ un tenseur constant, on en déduit que $\operatorname{div}\operatorname{div}\mathbf{M}_w \in \mathbb{P}^{k-2}(T) \subset \mathbb{P}^k(T)$, que $\mathbf{M}_w\mathbf{n}_{TF}\cdot\mathbf{n}_{TF} \in \mathbb{P}^k(F) \subset \mathbb{P}^{k+1}(F)$, et que $\operatorname{div}\mathbf{M}_w\cdot\mathbf{n}_{TF} + \partial_\tau(\mathbf{M}_w\mathbf{n}_{TF}\cdot\boldsymbol{\tau}) \in \mathbb{P}^{k-1}(F) \subset \mathbb{P}^k(F)$. Par conséquent, pour tout $w \in \mathbb{P}^{k+2}(T)$, on a

$$\begin{aligned} a_T(p^k_{\Delta,T}\underline{I}^k_T v, w) &= -(v, \operatorname{div}\operatorname{div}\mathbf{M}_w)_T - \sum_{F\in\mathcal{F}_T}(\partial_{n_{TF}} v, \mathbf{M}_w\mathbf{n}_{TF}\cdot\mathbf{n}_{TF})_F + \\ &\quad + \sum_{F\in\mathcal{F}_T}\big(v, \operatorname{div}\mathbf{M}_w\cdot\mathbf{n}_{TF} + \partial_\tau(\mathbf{M}_w\mathbf{n}_{TF}\cdot\boldsymbol{\tau})\big)_F = \\ &= a_T(v, w), \end{aligned}$$

où la dernière égalité est obtenue en utilisant les formules d'intégration par parties sur les termes du second membre, ainsi que la propriété de symétrie de $\mathbb{A}$. □

On peut donc appliquer le Lemme de Céa [15] (Chap. 2, § 2.4) à la forme bilinéaire $a_T(\cdot,\cdot)$, continue et coercive sur $H^2(T)/\mathbb{P}^1(T) \times H^2(T)/\mathbb{P}^1(T)$. On obtient

$$\|\nabla\nabla v - \nabla\nabla p^k_{\Delta,T}\underline{I}^k_T v\|_T \leqslant \sqrt{\frac{\mathcal{A}^+}{\mathcal{A}^-}} \inf_{z\in\mathbb{P}^{k+2}(T)/\mathbb{P}^1(T)} \|\nabla\nabla v - \nabla\nabla z\|_T,$$

où $\mathcal{A}^-$ et $\mathcal{A}^+$ sont des constantes positives telles que

$$\forall \mathbf{U} \in \mathrm{Sym}(2), \quad \mathcal{A}^-\|\mathbf{U}\|_2^2 \leqslant \mathbb{A}\mathbf{U}:\mathbf{U} \leqslant \mathcal{A}^+\|\mathbf{U}\|_2^2.$$

Grâce au Lemme 6.1, l'opérateur $p^k_{\Delta,T}\underline{I}^k_T$ possède des propriétés d'approximation optimales dans $\mathbb{P}^{k+2}(T)$; c'est à dire, grâce à l'estimation (6.4) avec $\ell = k+2$, $s = k+3$ et $m = 2$,

$$\|\nabla\nabla(p^k_{\Delta,T}\underline{I}^k_T v - v)\|_T \lesssim h_T^{k+1}\|v\|_{H^{k+3}(T)}, \quad \forall v \in H^{k+3}(T).$$

6.4 Problème Discret

On introduit maintenant l'*espace des DDL globaux*

$$\underline{U}^k_h \coloneqq \left\{ \bigtimes_{T\in\mathcal{T}_h} \mathbb{P}^k(T) \right\} \times \left\{ \bigtimes_{F\in\mathcal{F}_T} \mathbb{P}^{k+1}(F) \right\} \times \left\{ \bigtimes_{F\in\mathcal{F}_T} \mathbb{P}^k(F) \right\}, \tag{6.10}$$

l'espace associé caractérisant les conditions au bord

$$\begin{aligned} \underline{U}^k_{h,0} \coloneqq \big\{ \underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_{n,F})_{F\in\mathcal{F}_h}, (v_F)_{F\in\mathcal{F}_h}) \in \underline{U}^k_h : \\ v_F = v_{n,F} = 0 \text{ pour tout } F \in F^b_h \big\}, \end{aligned} \tag{6.11}$$

l'opérateur de *restriction* $\underline{L}_T : \underline{U}_h^k \to \underline{U}_T^k$ défini pour tout $T \in \mathcal{T}_h$, qui envoie les DDL globaux dans $\underline{U}_h^k$ sur les DDL locaux correspondants dans $\underline{U}_T^k$.

On introduit la forme bilinéaire globale sur $\underline{U}_h^k \times \underline{U}_h^k$:

$$\widehat{a}_h(\underline{u}_h, \underline{v}_h) \coloneqq \sum_{T \in \mathcal{T}_h} \widehat{a}_T(\underline{L}_T \underline{u}_h, \underline{L}_T \underline{v}_h),$$

où la forme bilinéaire locale $\widehat{a}_T$ sur $\underline{U}_T^k \times \underline{U}_T^k$ s'écrit sous la forme

$$\widehat{a}_T(\underline{u}_T, \underline{v}_T) = a_T(p_{\Delta,T}^k \underline{u}_T, p_{\Delta,T}^k \underline{v}_T) + s_T(\underline{u}_T, \underline{v}_T).$$

La forme bilinéaire $s_T(\cdot,\cdot)$, dite de *stabilisation locale*, prend en compte la non-conformité de la méthode. En effet, l'application

$$\underline{U}_T^k \times \underline{U}_T^k \ni (\underline{u}_T, \underline{v}_T) \mapsto a_T(p_{\Delta,T}^k \underline{u}_T, p_{\Delta,T}^k \underline{v}_T) \in \mathbb{R}$$

définit une forme bilinéaire continue *non coercive* sur l'espace discret $\underline{U}_T^k \times \underline{U}_T^k$: cette forme s'annule sur les éléments $\underline{u}_T$ tels que $p_{\Delta,T}^k \underline{u}_T \in \mathbb{P}^1(T)$. On définit $s_T(\cdot,\cdot)$ par

$$\begin{aligned} s_T(\underline{u}_T, \underline{v}_T) \ &= \sum_{F \in \mathcal{F}_T} h_F^{-1} \left(\pi_F^{k+1}(\partial_{n_{TF}} p_{\Delta,T}^k \underline{u}_T - u_{n,F}), \pi_F^{k+1}(\partial_{n_{TF}} p_{\Delta,T}^k \underline{v}_T - v_{n,F}) \right)_F + \\ &\quad + \sum_{F \in \mathcal{F}_T} h_F^{-3} \left(\pi_F^k(p_{\Delta,T}^k \underline{u}_T - u_F), \pi_F^k(p_{\Delta,T}^k \underline{v}_T - v_F) \right)_F + \\ &\quad + h_T^{-4} \left(\pi_T^k(p_{\Delta,T}^k \underline{u}_T - u_T), \pi_T^k(p_{\Delta,T}^k \underline{v}_T - v_T) \right)_T, \ \ \forall (\underline{u}_T, \underline{v}_T) \in \underline{U}_T^k \times \underline{U}_T^k. \end{aligned} \tag{6.12}$$

La forme linéaire dans (6.2) peut être discrétisée grâce à la forme linéaire sur $\underline{U}_h^k$

$$l_h(\underline{v}_h) \coloneqq \sum_{T \in \mathcal{T}_h} (f, v_T)_T.$$

Le *problème discret* est donc formulé comme suit :

$$\begin{gathered} \text{Trouver } \underline{u}_h \in \underline{U}_{h,0}^k \text{ tel que, pour tout } \underline{v}_h \in \underline{U}_{h,0}^k, \\ \widehat{a}_h(\underline{u}_h, \underline{v}_h) = l_h(\underline{v}_h). \end{gathered} \tag{6.13}$$

Par la suite, il faudra :

(a) démontrer que le problème (6.13) est *bien posé* ;

(b) démontrer que le problème (6.13) est *consistant* ;

(c) obtenir des *estimations d'erreur*.

La démonstration de ces propriétés dépend de la définition de la forme bilinéaire locale $\widehat{a}_T(\cdot,\cdot)$; en particulier, on doit montrer que $\widehat{a}_T(\cdot,\cdot)$ vérifie les propriétés de stabilité et continuité locales :

$$\begin{gathered} \exists \mu > 0 \text{ indépendant de } h \text{ tel que, pour tout } T \in \mathcal{T}_h \text{ et tout } \underline{v}_T \in \underline{U}_T^k, \text{ l'on ait} \\ \mu^{-1} \|\underline{v}_T\|_{2,T}^2 \leqslant \widehat{a}_T(\underline{v}_T, \underline{v}_T) \leqslant \mu \|\underline{v}_T\|_{2,T}^2, \end{gathered} \tag{6.14}$$

où $\|\cdot\|_{2,T}$ est la norme discrète locale

$$\|\underline{v}_T\|_{2,T}^2 \coloneqq \|\nabla\nabla v_T\|_T^2 + \sum_{F\in\mathcal{F}_T} h_F^{-1}\|v_{n,F} - \partial_{n_{TF}} v_T\|_F^2 + \sum_{F\in\mathcal{F}_T} h_F^{-3}\|v_F - v_T\|_F^2, \quad \forall \underline{v}_T \in \underline{U}_T^k.$$

Il faudra ensuite implémenter cette méthode dans un code afin de la comparer avec les résultats obtenus dans le chapitre précédent.

Conclusions

Nous avons présenté un modèle mathématique de structures de type capteur ou actionneur, caractérisées par des couplages linéaires multiphysiques (magnéto-électro-thermo-élastiques), en justifiant sa cohérence au sens de la thermodynamique des milieux continus. Nous avons montré que les problèmes dynamique et quasi-statique sont bien posés, le premier dans le cadre de la théorie de Hille-Yosida, le deuxième avec la méthode de Faedo-Galerkin. Nous avons fourni une première validation de l'hypothèse quasi-statique en effectuant une adimensionnalisation formelle sur les équations du problème dynamique.

Ensuite, à partir du problème quasi-statique, nous avons présenté un modèle bidimensionnel pour une structure en forme de plaque qui se comporte comme capteur et/ou actionneur. Le modèle a été déduit grâce à la méthode des développements asymptotiques, sous les hypothèses d'anisotropie et homogénéité, et en considérant quatre types différents de conditions au bord. Nous avons validé les résultats fournis par l'analyse asymptotique en montrant des théorèmes de convergence faible (Théorèmes 4.3, 4.5, 4.7 et 4.9). Chacun des quatre problèmes de plaque résultant de l'analyse asymptotique se découple en un problème de flexion et en un problème de membrane totalement ou partiellement couplé.

Nous avons enfin concentré notre attention sur le problème de flexion, qui tient en compte un effet d'inertie de rotation, caractérisant tous les quatres problèmes de plaque et se présentant toujours sous la même forme. Nous avons présenté une étude mathématique et numérique de ce problème, et l'analyse numérique a été validée avec des tests effectués sous l'environnement `FreeFEM++`, en utilisant la méthode de Newmark du point milieu combiné avec une discrétisation conforme en espace, caractérisée par des éléments finis HCT.

Concernant des perspectives futures de recherche portant sur les aspects mathématiques, on mentionne d'abord la justification rigoureuse de la convergence de la solution du problème dynamique vers celle du problème quasi-statique lorsque $\delta \to 0$. En effet, le problème dynamique a été étudié dans le cadre de la théorie de Hille-Yosida, en obtenant ainsi une solution régulière en temps, tandis que la solution du problème quasi-statique a été obtenue dans une forme faible grâce à la méthode de Faedo-Galerkin. La difficulté rencontrée pour ce qui concerne l'étude du problème quasi-statique dans le cadre de la théorie des semi-groupes est que, à la différence du terme de couplage thermo-élastique $\boldsymbol{\beta} : \mathbf{e}(\dot{\mathbf{u}})$, les deux termes de couplage pyroélectrique et pyromagnétique $\mathbf{p} \cdot \nabla\dot{\varphi}$ et $\mathbf{m} \cdot \nabla\dot{\zeta}$ présents dans la dernière équation du système (bilan de l'énergie) n'ont pas de "contrepartie symétrique" dans la première équation du système (bilan de la quantité de mouvement).

Une direction de recherche concernant les applications peut être l'étude d'une structure laminée, de type plaque ou coque, contenant une couche de matériau METE ; le cas d'une couche purement piézoélectrique a été traité, e.g., dans [60] et [61].

Enfin, pour ce qui concerne les aspects numériques, comme des éléments finis de classe C^1 sont chers en termes de calculs pour le traitement de problèmes de plaque en flexion, il est intéressant d'utiliser une méthode de discrétisation en espace de type non conforme. Nous avons présenté de manière succincte, dans le dernier chapitre, une première approche de la version statique du problème de flexion avec une méthode hybride et d'ordre élevé. Une fois l'analyse numérique complétée dans le cas statique, afin d'étudier le problème dynamique avec inertie de rotation, il sera nécessaire de coupler une telle discrétisation en espace avec une discrétisation en temps, par exemple à nouveau de type Newmark. L'étape finale de cette procédure consiste, clairement, à implémenter la nouvelle méthode numérique de façon efficace.

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