TEMPERATURE INFLUENCE ON SMART STRUCTURES: A FIRST APPROACH

F. Bonaldi[1], G. Geymonat[2], F. Krasucki[1], M. Serpilli[3]

[1]Institut de Mathématiques et de Modélisation de Montpellier, Université Montpellier 2
[2]Laboratoire de Mécanique des Solides, École Polytechnique
[3]Department of Civil and Building Engineering, and Architecture, Polytechnic University of Marche

11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)

WCCM XI - ECCM V - ECFD VI
BARCELONA 2014

Barcelona, July 22nd, 2014

Introduction

- Smart structures control their strain state through **sensors** and **actuators**, embedded within them in the form of **thermo-electromagnetoelastic material** layers. Applications are found, e.g., in aircraft structures, health monitoring, vibration and shape control of flexible structures.

Introduction

- Smart structures control their strain state through **sensors** and **actuators**, embedded within them in the form of **thermo-electromagnetoelastic material** layers. Applications are found, e.g., in aircraft structures, health monitoring, vibration and shape control of flexible structures.
- Classical models of sensors and actuators proposed in literature do not account for temperature influence, neither for possible magnetic effects (as, e.g., in a $BaTiO_3$-$CoFe_2O_4$ complex [Kondaiah et al., 2013]).

Introduction

- Smart structures control their strain state through **sensors** and **actuators**, embedded within them in the form of **thermo-electromagnetoelastic material** layers. Applications are found, e.g., in aircraft structures, health monitoring, vibration and shape control of flexible structures.
- Classical models of sensors and actuators proposed in literature do not account for temperature influence, neither for possible magnetic effects (as, e.g., in a $BaTiO_3$-$CoFe_2O_4$ complex [Kondaiah et al., 2013]).

$\Longrightarrow$

Outline

- Enrichment of the model by adding the energy balance equation, so as to account for temperature influence.

Introduction

- Smart structures control their strain state through **sensors** and **actuators**, embedded within them in the form of **thermo-electromagnetoelastic material** layers. Applications are found, e.g., in aircraft structures, health monitoring, vibration and shape control of flexible structures.
- Classical models of sensors and actuators proposed in literature do not account for temperature influence, neither for possible magnetic effects (as, e.g., in a $BaTiO_3$-$CoFe_2O_4$ complex [Kondaiah et al., 2013]).

$\Longrightarrow$

Outline

- Enrichment of the model by adding the energy balance equation, so as to account for temperature influence.
- Validation of the typical **quasi-static** assumption, by carrying out a formal nondimensionalization of the equations.

Introduction

- Smart structures control their strain state through **sensors** and **actuators**, embedded within them in the form of **thermo-electromagnetoelastic material** layers. Applications are found, e.g., in aircraft structures, health monitoring, vibration and shape control of flexible structures.
- Classical models of sensors and actuators proposed in literature do not account for temperature influence, neither for possible magnetic effects (as, e.g., in a $BaTiO_3$-$CoFe_2O_4$ complex [Kondaiah et al., 2013]).

$\Longrightarrow$

Outline

- Enrichment of the model by adding the energy balance equation, so as to account for temperature influence.
- Validation of the typical **quasi-static** assumption, by carrying out a formal nondimensionalization of the equations.
- Deduction of a thin thermo-electromagnetoelastic plate model, based on the quasi-static 3D model, by means of the asymptotic expansions method.

Problem Statement

Let $\Omega \subset \mathbb{R}^3$ be an open bounded region and $\widehat{\mathcal{X}} := (\mathbf{u}, \mathbf{E}, \mathbf{H}, \theta)$.

Field Equations

$$\begin{cases} \rho\ddot{\mathbf{u}} - \operatorname{div}\boldsymbol{\sigma}(\widehat{\mathcal{X}}) = \mathbf{f} & \mathbf{x} \in \Omega,\ t > 0, \\ \operatorname{div}\mathbf{D}(\widehat{\mathcal{X}}) = 0 & \mathbf{x} \in \Omega,\ t > 0, \\ \operatorname{div}\mathbf{B}(\widehat{\mathcal{X}}) = 0 & \mathbf{x} \in \Omega,\ t > 0, \\ \dot{\mathbf{D}}(\widehat{\mathcal{X}}) - \nabla \times \mathbf{H} = \mathbf{0} & \mathbf{x} \in \Omega,\ t > 0, \\ \dot{\mathbf{B}}(\widehat{\mathcal{X}}) + \nabla \times \mathbf{E} = \mathbf{0} & \mathbf{x} \in \Omega,\ t > 0, \\ \dot{\mathcal{S}}(\widehat{\mathcal{X}}) + \dfrac{1}{T_0}\operatorname{div}\mathbf{q}(\theta) = r & \mathbf{x} \in \Omega,\ t > 0. \end{cases}$$

Problem Statement

Let $\Omega \subset \mathbb{R}^3$ be an open bounded region and $\widehat{\mathcal{X}} := (\mathbf{u}, \mathbf{E}, \mathbf{H}, \theta)$.

Field Equations

$$
\begin{cases}
\rho\ddot{\mathbf{u}} - \operatorname{div}\boldsymbol{\sigma}(\widehat{\mathcal{X}}) = \mathbf{f} & \mathbf{x} \in \Omega,\ t > 0, \\
\operatorname{div}\mathbf{D}(\widehat{\mathcal{X}}) = 0 & \mathbf{x} \in \Omega,\ t > 0, \\
\operatorname{div}\mathbf{B}(\widehat{\mathcal{X}}) = 0 & \mathbf{x} \in \Omega,\ t > 0, \\
\dot{\mathbf{D}}(\widehat{\mathcal{X}}) - \nabla \times \mathbf{H} = \mathbf{0} & \mathbf{x} \in \Omega,\ t > 0, \\
\dot{\mathbf{B}}(\widehat{\mathcal{X}}) + \nabla \times \mathbf{E} = \mathbf{0} & \mathbf{x} \in \Omega,\ t > 0, \\
\dot{\mathcal{S}}(\widehat{\mathcal{X}}) + \dfrac{1}{T_0}\operatorname{div}\mathbf{q}(\theta) = r & \mathbf{x} \in \Omega,\ t > 0.
\end{cases}
$$

Constitutive Assumptions

$$
\begin{aligned}
&\boldsymbol{\sigma}(\widehat{\mathcal{X}}) = \mathbf{Ce}(\mathbf{u}) - \mathbf{PE} - \mathbf{RH} - \boldsymbol{\beta}\theta, \\
&\mathbf{D}(\widehat{\mathcal{X}}) = \mathbf{P}^T\mathbf{e}(\mathbf{u}) + \mathbf{XE} + \boldsymbol{\alpha}\mathbf{H} + \mathbf{p}\theta, \\
&\mathbf{B}(\widehat{\mathcal{X}}) = \mathbf{R}^T\mathbf{e}(\mathbf{u}) + \boldsymbol{\alpha}\mathbf{E} + \mathbf{MH} + \mathbf{m}\theta, \\
&\mathcal{S}(\widehat{\mathcal{X}}) = \boldsymbol{\beta} : \mathbf{e}(\mathbf{u}) + \mathbf{p} \cdot \mathbf{E} + \mathbf{m} \cdot \mathbf{H} + c_v\theta, \\
&\mathbf{q}(\theta) = -\mathbf{Q}\nabla\theta.
\end{aligned}
$$

Problem Statement

Let $\Omega \subset \mathbb{R}^3$ be an open bounded region and $\widehat{\mathcal{X}} := (\mathbf{u}, \mathbf{E}, \mathbf{H}, \theta)$.

Field Equations

$$\begin{cases} \rho\ddot{\mathbf{u}} - \operatorname{div}\boldsymbol{\sigma}(\widehat{\mathcal{X}}) = \mathbf{f} & \mathbf{x} \in \Omega,\ t > 0, \\ \operatorname{div}\mathbf{D}(\widehat{\mathcal{X}}) = 0 & \mathbf{x} \in \Omega,\ t > 0, \\ \operatorname{div}\mathbf{B}(\widehat{\mathcal{X}}) = 0 & \mathbf{x} \in \Omega,\ t > 0, \\ \dot{\mathbf{D}}(\widehat{\mathcal{X}}) - \nabla \times \mathbf{H} = \mathbf{0} & \mathbf{x} \in \Omega,\ t > 0, \\ \dot{\mathbf{B}}(\widehat{\mathcal{X}}) + \nabla \times \mathbf{E} = \mathbf{0} & \mathbf{x} \in \Omega,\ t > 0, \\ \dot{\mathcal{S}}(\widehat{\mathcal{X}}) + \dfrac{1}{T_0}\operatorname{div}\mathbf{q}(\theta) = r & \mathbf{x} \in \Omega,\ t > 0. \end{cases}$$

Constitutive Assumptions

$$\boldsymbol{\sigma}(\widehat{\mathcal{X}}) = \mathbf{C}\mathbf{e}(\mathbf{u}) - \mathbf{P}\mathbf{E} - \mathbf{R}\mathbf{H} - \boldsymbol{\beta}\theta,$$
$$\mathbf{D}(\widehat{\mathcal{X}}) = \mathbf{P}^T\mathbf{e}(\mathbf{u}) + \mathbf{X}\mathbf{E} + \boldsymbol{\alpha}\mathbf{H} + \mathbf{p}\theta,$$
$$\mathbf{B}(\widehat{\mathcal{X}}) = \mathbf{R}^T\mathbf{e}(\mathbf{u}) + \boldsymbol{\alpha}\mathbf{E} + \mathbf{M}\mathbf{H} + \mathbf{m}\theta,$$
$$\mathcal{S}(\widehat{\mathcal{X}}) = \boldsymbol{\beta} : \mathbf{e}(\mathbf{u}) + \mathbf{p}\cdot\mathbf{E} + \mathbf{m}\cdot\mathbf{H} + c_v\theta,$$
$$\mathbf{q}(\theta) = -\mathbf{Q}\nabla\theta.$$

Initial Conditions

$$\begin{cases} \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \\ \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{u}_1(\mathbf{x}), \\ \mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}), \\ \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}), \\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}). \end{cases}$$

Boundary Conditions

$$\begin{cases} \boldsymbol{\sigma}(\widehat{\mathcal{X}})\mathbf{n} = \mathbf{g} & \text{on } \partial\Omega_1 \times (0,t_0), & \mathbf{u} = \overline{\mathbf{u}} & \text{on } \partial\Omega_2 \times (0,t_0), \\ \mathbf{D}(\widehat{\mathcal{X}})\cdot\mathbf{n} = d & \text{on } \partial\Omega_1 \times (0,t_0), & \mathbb{T}\mathbf{E} = \overline{\mathbf{E}} & \text{on } \partial\Omega_2 \times (0,t_0), \\ \mathbf{B}(\widehat{\mathcal{X}})\cdot\mathbf{n} = b & \text{on } \partial\Omega_1 \times (0,t_0), & \mathbb{T}\mathbf{H} = \overline{\mathbf{H}} & \text{on } \partial\Omega_2 \times (0,t_0), \\ -\mathbf{q}(\theta)\cdot\mathbf{n} = \varrho & \text{on } \partial\Omega_1 \times (0,t_0), & \theta = \overline{\theta} & \text{on } \partial\Omega_2 \times (0,t_0). \end{cases}$$

Nondimensionalization

By carrying out a formal nondimensionalization of the evolution field equations, with an appropriate choice of the units of measurement of $\mathbf{E}$ and $\mathbf{H}$, we get two expressions of the form

$$\nabla \times \mathbf{E}_r = -\delta \left(\mathbf{M}_r \dot{\mathbf{H}}_r + \kappa \, \mathbf{R}_r^T \mathbf{e}(\dot{\mathbf{u}}_r) + \alpha_+ c_0 \, \boldsymbol{\alpha}_r \dot{\mathbf{E}}_r + \upsilon \, \mathbf{m}_r \dot{\theta}_r \right),$$
$$\nabla \times \mathbf{H}_r = \delta \left(\mathbf{X}_r \dot{\mathbf{E}}_r + \chi \, \mathbf{P}_r^T \mathbf{e}(\dot{\mathbf{u}}_r) + \alpha_+ c_0 \, \boldsymbol{\alpha}_r \dot{\mathbf{H}}_r + \varsigma \, \mathbf{p}_r \dot{\theta}_r \right),$$

with $\delta \simeq 10^{-5}$.

Nondimensionalization

By carrying out a formal nondimensionalization of the evolution field equations, with an appropriate choice of the units of measurement of $\mathbf{E}$ and $\mathbf{H}$, we get two expressions of the form

$$\nabla \times \mathbf{E}_r = -\delta \left(\mathbf{M}_r \dot{\mathbf{H}}_r + \kappa \, \mathbf{R}_r^T \mathbf{e}(\dot{\mathbf{u}}_r) + \alpha_+ c_0 \, \boldsymbol{\alpha}_r \dot{\mathbf{E}}_r + \upsilon \, \mathbf{m}_r \dot{\theta}_r \right),$$
$$\nabla \times \mathbf{H}_r = \delta \left(\mathbf{X}_r \dot{\mathbf{E}}_r + \chi \, \mathbf{P}_r^T \mathbf{e}(\dot{\mathbf{u}}_r) + \alpha_+ c_0 \, \boldsymbol{\alpha}_r \dot{\mathbf{H}}_r + \varsigma \, \mathbf{p}_r \dot{\theta}_r \right),$$

with $\delta \simeq 10^{-5}$.

- In the limit $\delta \to 0$, if the time derivatives on the right-hand sides are bounded, we get

Quasi-Static Assumption

$$\nabla \times \mathbf{E}_r = \mathbf{0} \iff \mathbf{E}_r = -\nabla \varphi_r$$
$$\nabla \times \mathbf{H}_r = \mathbf{0} \iff \mathbf{H}_r = -\nabla \zeta_r$$

Quasi-Static Problem for a Plate-Like Body

We now identify Ω with a plate-like region Ω^ε of thickness $2\varepsilon h$. Let $\mathcal{X}^\varepsilon := (\mathbf{u}^\varepsilon, \varphi^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon)$. The field equations become

Quasi-Static System

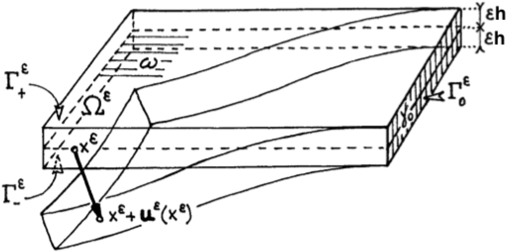
$$\begin{cases} \rho^\varepsilon \ddot{\mathbf{u}}^\varepsilon - \mathrm{div}^\varepsilon \boldsymbol{\sigma}^\varepsilon(\mathcal{X}^\varepsilon) = \mathbf{f}^\varepsilon & \mathbf{x} \in \Omega^\varepsilon,\, t > 0, \\ \mathrm{div}^\varepsilon \mathbf{D}^\varepsilon(\mathcal{X}^\varepsilon) = 0 & \mathbf{x} \in \Omega^\varepsilon,\, t > 0, \\ \mathrm{div}^\varepsilon \mathbf{B}^\varepsilon(\mathcal{X}^\varepsilon) = 0 & \mathbf{x} \in \Omega^\varepsilon,\, t > 0, \\ \dot{\mathcal{S}}^\varepsilon(\mathcal{X}^\varepsilon) + \dfrac{1}{T_0}\mathrm{div}^\varepsilon \mathbf{q}^\varepsilon(\theta^\varepsilon) = r^\varepsilon & \mathbf{x} \in \Omega^\varepsilon,\, t > 0. \end{cases}$$

Quasi-Static Problem for a Plate-Like Body

We now identify Ω with a plate-like region Ω^ε of thickness $2\varepsilon h$. Let $\mathcal{X}^\varepsilon := (\mathbf{u}^\varepsilon, \varphi^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon)$. The field equations become

Quasi-Static System

$$\begin{cases} \rho^\varepsilon \ddot{\mathbf{u}}^\varepsilon - \mathrm{div}^\varepsilon \boldsymbol{\sigma}^\varepsilon(\mathcal{X}^\varepsilon) = \mathbf{f}^\varepsilon & \mathbf{x} \in \Omega^\varepsilon,\ t > 0, \\ \mathrm{div}^\varepsilon \mathbf{D}^\varepsilon(\mathcal{X}^\varepsilon) = 0 & \mathbf{x} \in \Omega^\varepsilon,\ t > 0, \\ \mathrm{div}^\varepsilon \mathbf{B}^\varepsilon(\mathcal{X}^\varepsilon) = 0 & \mathbf{x} \in \Omega^\varepsilon,\ t > 0, \\ \dot{\mathcal{S}}^\varepsilon(\mathcal{X}^\varepsilon) + \dfrac{1}{T_0} \mathrm{div}^\varepsilon \mathbf{q}^\varepsilon(\theta^\varepsilon) = r^\varepsilon & \mathbf{x} \in \Omega^\varepsilon,\ t > 0. \end{cases}$$



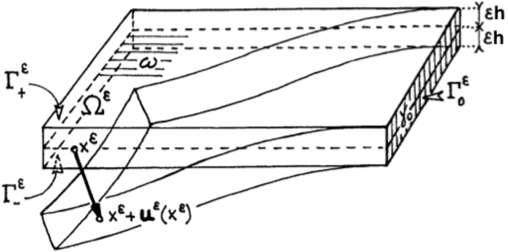
$$\Omega^\varepsilon := \omega \times (-\varepsilon h, \varepsilon h),$$
$$\partial\Omega^\varepsilon = \Gamma^\varepsilon \cup \Gamma^\varepsilon_\pm,$$
$$\Gamma^\varepsilon := \gamma \times (-\varepsilon h, \varepsilon h),\ \ \gamma := \partial\omega,$$
$$\Gamma^\varepsilon_\pm := \omega \times \{\pm\varepsilon h\}.$$

Quasi-Static Problem for a Plate-Like Body

We now identify Ω with a plate-like region Ω^ε of thickness $2\varepsilon h$. Let $\mathcal{X}^\varepsilon := (\mathbf{u}^\varepsilon, \varphi^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon)$. The field equations become

Quasi-Static System

$$\begin{cases} \rho^\varepsilon \ddot{\mathbf{u}}^\varepsilon - \operatorname{div}^\varepsilon \boldsymbol{\sigma}^\varepsilon(\mathcal{X}^\varepsilon) = \mathbf{f}^\varepsilon & \mathbf{x} \in \Omega^\varepsilon,\ t > 0, \\ \operatorname{div}^\varepsilon \mathbf{D}^\varepsilon(\mathcal{X}^\varepsilon) = 0 & \mathbf{x} \in \Omega^\varepsilon,\ t > 0, \\ \operatorname{div}^\varepsilon \mathbf{B}^\varepsilon(\mathcal{X}^\varepsilon) = 0 & \mathbf{x} \in \Omega^\varepsilon,\ t > 0, \\ \dot{\mathcal{S}}^\varepsilon(\mathcal{X}^\varepsilon) + \dfrac{1}{T_0} \operatorname{div}^\varepsilon \mathbf{q}^\varepsilon(\theta^\varepsilon) = r^\varepsilon & \mathbf{x} \in \Omega^\varepsilon,\ t > 0. \end{cases}$$



$\Omega^\varepsilon := \omega \times (-\varepsilon h, \varepsilon h)$,
$\partial\Omega^\varepsilon = \Gamma^\varepsilon \cup \Gamma^\varepsilon_\pm$,
$\Gamma^\varepsilon := \gamma \times (-\varepsilon h, \varepsilon h)$, $\gamma := \partial\omega$,
$\Gamma^\varepsilon_\pm := \omega \times \{\pm\varepsilon h\}$.

$\Gamma^\varepsilon = \Gamma^\varepsilon_0 \cup \Gamma^\varepsilon_1$, $\Gamma^\varepsilon_0 := \gamma_0 \times (-\varepsilon h, \varepsilon h)$ and $\Gamma^\varepsilon_1 := \gamma_1 \times (-\varepsilon h, \varepsilon h)$.
Let also $\widehat{\Gamma}^\varepsilon := \Gamma_\pm \cup \Gamma^\varepsilon_1$.

Initial and Boundary Conditions

Initial Conditions

$$\begin{cases} \mathbf{u}^{\varepsilon}(\mathbf{x}^{\varepsilon},0) = \mathbf{u}_0^{\varepsilon}(\mathbf{x}^{\varepsilon}) & \text{in } \Omega^{\varepsilon}, \\ \dot{\mathbf{u}}^{\varepsilon}(\mathbf{x}^{\varepsilon},0) = \mathbf{u}_1^{\varepsilon}(\mathbf{x}^{\varepsilon}) & \text{in } \Omega^{\varepsilon}, \\ \theta^{\varepsilon}(\mathbf{x}^{\varepsilon},0) = \theta_0^{\varepsilon}(\mathbf{x}^{\varepsilon}) & \text{in } \Omega^{\varepsilon}. \end{cases}$$

Initial and Boundary Conditions

Initial Conditions

$$\begin{cases} \mathbf{u}^\varepsilon(\mathbf{x}^\varepsilon, 0) = \mathbf{u}_0^\varepsilon(\mathbf{x}^\varepsilon) & \text{in } \Omega^\varepsilon, \\ \dot{\mathbf{u}}^\varepsilon(\mathbf{x}^\varepsilon, 0) = \mathbf{u}_1^\varepsilon(\mathbf{x}^\varepsilon) & \text{in } \Omega^\varepsilon, \\ \theta^\varepsilon(\mathbf{x}^\varepsilon, 0) = \theta_0^\varepsilon(\mathbf{x}^\varepsilon) & \text{in } \Omega^\varepsilon. \end{cases}$$

Thermo-mechanical Boundary Conditions

$$\begin{cases} \mathbf{u}^\varepsilon = \mathbf{0} & \text{on } \Gamma_0^\varepsilon, & \theta^\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ \boldsymbol{\sigma}^\varepsilon(\mathcal{X}^\varepsilon)\mathbf{n}^\varepsilon = \mathbf{g}^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon, & -\mathbf{q}^\varepsilon(\theta^\varepsilon) \cdot \mathbf{n}^\varepsilon = \varrho^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon. \end{cases}$$

Initial and Boundary Conditions

Initial Conditions

$$\begin{cases} \mathbf{u}^\varepsilon(\mathbf{x}^\varepsilon,0) = \mathbf{u}_0^\varepsilon(\mathbf{x}^\varepsilon) & \text{in } \Omega^\varepsilon, \\ \dot{\mathbf{u}}^\varepsilon(\mathbf{x}^\varepsilon,0) = \mathbf{u}_1^\varepsilon(\mathbf{x}^\varepsilon) & \text{in } \Omega^\varepsilon, \\ \theta^\varepsilon(\mathbf{x}^\varepsilon,0) = \theta_0^\varepsilon(\mathbf{x}^\varepsilon) & \text{in } \Omega^\varepsilon. \end{cases}$$

Thermo-mechanical Boundary Conditions

$$\begin{cases} \mathbf{u}^\varepsilon = \mathbf{0} & \text{on } \Gamma_0^\varepsilon, & \theta^\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ \boldsymbol{\sigma}^\varepsilon(\mathcal{X}^\varepsilon)\mathbf{n}^\varepsilon = \mathbf{g}^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon, & -\mathbf{q}^\varepsilon(\theta^\varepsilon)\cdot\mathbf{n}^\varepsilon = \varrho^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon. \end{cases}$$

Electromagnetic Boundary Conditions

$$\begin{cases} \varphi^\varepsilon = 0 & \text{on } \Gamma^\varepsilon, & \zeta^\varepsilon = \zeta^{\pm,\varepsilon} & \text{on } \Gamma_\pm^\varepsilon, \\ \mathbf{D}^\varepsilon(\mathcal{X}^\varepsilon)\cdot\mathbf{n}^\varepsilon = d^\varepsilon & \text{on } \Gamma_\pm^\varepsilon, & \mathbf{B}^\varepsilon(\mathcal{X}^\varepsilon)\cdot\mathbf{n}^\varepsilon = 0 & \text{on } \Gamma^\varepsilon. \end{cases}$$

Initial and Boundary Conditions

Initial Conditions

$$\begin{cases} \mathbf{u}^\varepsilon(\mathbf{x}^\varepsilon,0) = \mathbf{u}_0^\varepsilon(\mathbf{x}^\varepsilon) & \text{in } \Omega^\varepsilon, \\ \dot{\mathbf{u}}^\varepsilon(\mathbf{x}^\varepsilon,0) = \mathbf{u}_1^\varepsilon(\mathbf{x}^\varepsilon) & \text{in } \Omega^\varepsilon, \\ \theta^\varepsilon(\mathbf{x}^\varepsilon,0) = \theta_0^\varepsilon(\mathbf{x}^\varepsilon) & \text{in } \Omega^\varepsilon. \end{cases}$$

Thermo-mechanical Boundary Conditions

$$\begin{cases} \mathbf{u}^\varepsilon = \mathbf{0} & \text{on } \Gamma_0^\varepsilon, & \theta^\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ \boldsymbol{\sigma}^\varepsilon(\mathcal{X}^\varepsilon)\mathbf{n}^\varepsilon = \mathbf{g}^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon, & -\mathbf{q}^\varepsilon(\theta^\varepsilon)\cdot\mathbf{n}^\varepsilon = \varrho^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon. \end{cases}$$

Electromagnetic Boundary Conditions

$$\begin{cases} \varphi^\varepsilon = 0 & \text{on } \Gamma^\varepsilon, & \zeta^\varepsilon = \zeta^{\pm,\varepsilon} & \text{on } \Gamma_\pm^\varepsilon, \\ \mathbf{D}^\varepsilon(\mathcal{X}^\varepsilon)\cdot\mathbf{n}^\varepsilon = d^\varepsilon & \text{on } \Gamma_\pm^\varepsilon, & \mathbf{B}^\varepsilon(\mathcal{X}^\varepsilon)\cdot\mathbf{n}^\varepsilon = 0 & \text{on } \Gamma^\varepsilon. \end{cases}$$

- Our electromagnetic boundary conditions correspond to the case of a plate behaving *simultaneously* as a **piezoelectric sensor** and as a **piezomagnetic actuator**.
- By varying electromagnetic boundary conditions, one can reproduce a sensor-like, actuator-like or mixed behavior of the plate as a 3D body [Licht & Weller, 2010].

Asymptotic Analysis

Assumptions

- Displacement and temperature scalings:

$$u^\varepsilon_\alpha(\mathbf{x}^\varepsilon, t) = u_\alpha(\varepsilon)(\mathbf{x}, t),\ u^\varepsilon_3(\mathbf{x}^\varepsilon, t) = \varepsilon^{-1} u_3(\varepsilon)(\mathbf{x}, t),$$
$$\theta^\varepsilon(\mathbf{x}^\varepsilon, t) = \theta(\varepsilon)(\mathbf{x}, t), \quad \forall \mathbf{x}^\varepsilon = \pi^\varepsilon \mathbf{x} \in \overline{\Omega}^\varepsilon,\ t > 0.$$

- In order for the 2D plate model to reproduce the desired electromagnetic behavior, precise scaling assumptions on φ and ζ must be made [Licht & Weller, 2007]. In the case of the **piezoelectric sensor - piezomagnetic actuator** problem, we have

$$\varphi^\varepsilon(\mathbf{x}^\varepsilon, t) = \varphi(\varepsilon)(\mathbf{x}, t), \quad \zeta^\varepsilon(\mathbf{x}^\varepsilon, t) = \varepsilon\zeta(\varepsilon)(\mathbf{x}, t), \quad \forall \mathbf{x}^\varepsilon = \pi^\varepsilon \mathbf{x} \in \overline{\Omega}^\varepsilon,\ t > 0.$$

Asymptotic Analysis

Assumptions

- Displacement and temperature scalings:

$$u_\alpha^\varepsilon(\mathbf{x}^\varepsilon, t) = u_\alpha(\varepsilon)(\mathbf{x}, t),\ u_3^\varepsilon(\mathbf{x}^\varepsilon, t) = \varepsilon^{-1} u_3(\varepsilon)(\mathbf{x}, t),$$
$$\theta^\varepsilon(\mathbf{x}^\varepsilon, t) = \theta(\varepsilon)(\mathbf{x}, t), \quad \forall \mathbf{x}^\varepsilon = \pi^\varepsilon \mathbf{x} \in \overline{\Omega}^\varepsilon,\ t > 0.$$

- In order for the 2D plate model to reproduce the desired electromagnetic behavior, precise scaling assumptions on φ and ζ must be made [Licht & Weller, 2007]. In the case of the **piezoelectric sensor - piezomagnetic actuator** problem, we have

$$\varphi^\varepsilon(\mathbf{x}^\varepsilon, t) = \varphi(\varepsilon)(\mathbf{x}, t), \quad \zeta^\varepsilon(\mathbf{x}^\varepsilon, t) = \varepsilon\zeta(\varepsilon)(\mathbf{x}, t), \quad \forall \mathbf{x}^\varepsilon = \pi^\varepsilon \mathbf{x} \in \overline{\Omega}^\varepsilon,\ t > 0.$$

- In general, the desired electromagnetic behavior of the plate is obtained by varying the powers of ε in the scalings of φ and ζ [Licht & Weller, 2007]:

$$\varphi^\varepsilon(\mathbf{x}^\varepsilon) = \varepsilon^p \varphi(\varepsilon)(\mathbf{x}), \quad \zeta^\varepsilon(\mathbf{x}^\varepsilon) = \varepsilon^q \zeta(\varepsilon)(\mathbf{x}), \quad p, q \in \{0, 1\}.$$

Asymptotic Analysis

Assumptions

- Displacement and temperature scalings:

$$u_\alpha^\varepsilon(\mathbf{x}^\varepsilon, t) = u_\alpha(\varepsilon)(\mathbf{x}, t),\ u_3^\varepsilon(\mathbf{x}^\varepsilon, t) = \varepsilon^{-1} u_3(\varepsilon)(\mathbf{x}, t),$$
$$\theta^\varepsilon(\mathbf{x}^\varepsilon, t) = \theta(\varepsilon)(\mathbf{x}, t), \quad \forall \mathbf{x}^\varepsilon = \pi^\varepsilon \mathbf{x} \in \overline{\Omega}^\varepsilon,\ t > 0.$$

- In order for the 2D plate model to reproduce the desired electromagnetic behavior, precise scaling assumptions on φ and ζ must be made [Licht & Weller, 2007]. In the case of the **piezoelectric sensor - piezomagnetic actuator** problem, we have

$$\varphi^\varepsilon(\mathbf{x}^\varepsilon, t) = \varphi(\varepsilon)(\mathbf{x}, t), \quad \zeta^\varepsilon(\mathbf{x}^\varepsilon, t) = \varepsilon \zeta(\varepsilon)(\mathbf{x}, t), \quad \forall \mathbf{x}^\varepsilon = \pi^\varepsilon \mathbf{x} \in \overline{\Omega}^\varepsilon,\ t > 0.$$

- In general, the desired electromagnetic behavior of the plate is obtained by varying the powers of ε in the scalings of φ and ζ [Licht & Weller, 2007]:

$$\varphi^\varepsilon(\mathbf{x}^\varepsilon) = \varepsilon^p \varphi(\varepsilon)(\mathbf{x}), \quad \zeta^\varepsilon(\mathbf{x}^\varepsilon) = \varepsilon^q \zeta(\varepsilon)(\mathbf{x}), \quad p, q \in \{0, 1\}.$$

- Thus, boundary conditions **and** scaling assumptions **both** play a crucial role in the determination of a 2D electromagnetic plate model.

Asymptotic Analysis

Results

- The limit displacement field satisfies the Kirchhoff-Love kinematical assumptions:

$$\widetilde{\mathbf{u}}^0(\tilde{\mathbf{x}}, x_3) = \mathbf{u}_H(\tilde{\mathbf{x}}) - x_3 \nabla_\tau w(\tilde{\mathbf{x}}) \quad \text{and} \quad u_3^0(\tilde{\mathbf{x}}, x_3) = w(\tilde{\mathbf{x}}).$$

Asymptotic Analysis

Results

- The limit displacement field satisfies the Kirchhoff-Love kinematical assumptions:

$$\widetilde{\mathbf{u}}^0(\tilde{\mathbf{x}}, x_3) = \mathbf{u}_H(\tilde{\mathbf{x}}) - x_3 \nabla_\tau w(\tilde{\mathbf{x}}) \quad \text{and} \quad u_3^0(\tilde{\mathbf{x}}, x_3) = w(\tilde{\mathbf{x}}).$$

- The limit temperature variation is independent of x_3: $\theta^0(\tilde{\mathbf{x}}, x_3) = \vartheta(\tilde{\mathbf{x}})$.

Asymptotic Analysis

Results

- The limit displacement field satisfies the Kirchhoff-Love kinematical assumptions:

$$\widetilde{\mathbf{u}}^0(\tilde{\mathbf{x}}, x_3) = \mathbf{u}_H(\tilde{\mathbf{x}}) - x_3 \nabla_\tau w(\tilde{\mathbf{x}}) \quad \text{and} \quad u_3^0(\tilde{\mathbf{x}}, x_3) = w(\tilde{\mathbf{x}}).$$

- The limit temperature variation is independent of x_3: $\theta^0(\tilde{\mathbf{x}}, x_3) = \vartheta(\tilde{\mathbf{x}})$.
- The limit electric potential is independent of x_3: $\varphi^0(\tilde{\mathbf{x}}, x_3) = \phi(\tilde{\mathbf{x}})$.

Asymptotic Analysis

Results

- The limit displacement field satisfies the Kirchhoff-Love kinematical assumptions:

$$\widetilde{\mathbf{u}}^0(\tilde{\mathbf{x}}, x_3) = \mathbf{u}_H(\tilde{\mathbf{x}}) - x_3 \nabla_\tau w(\tilde{\mathbf{x}}) \quad \text{and} \quad u_3^0(\tilde{\mathbf{x}}, x_3) = w(\tilde{\mathbf{x}}).$$

- The limit temperature variation is independent of x_3: $\theta^0(\tilde{\mathbf{x}}, x_3) = \vartheta(\tilde{\mathbf{x}})$.
- The limit electric potential is independent of x_3: $\varphi^0(\tilde{\mathbf{x}}, x_3) = \phi(\tilde{\mathbf{x}})$.
- The limit magnetic potential is a **quadratic** function of x_3:

$$\zeta^0(\tilde{\mathbf{x}}, x_3) = \sum_{k=0}^{2} z^k(\tilde{\mathbf{x}}) x_3^k,$$

$$z^0 := \frac{\zeta^+ + \zeta^-}{2} + \frac{h^2}{2} \widetilde{\mathbf{\Lambda}} : \nabla_\tau \nabla_\tau w, \quad z^1 := \frac{\zeta^+ - \zeta^-}{2h}, \quad z^2 := -\frac{1}{2} \widetilde{\mathbf{\Lambda}} : \nabla_\tau \nabla_\tau w, \quad \widetilde{\mathbf{\Lambda}} := \frac{\widetilde{\mathbf{R}}_3}{\widetilde{M}_{33}}.$$

Limit Evolution Problems

- The limit problem decouples into two evolution subproblems.

Limit Evolution Problems

- The limit problem decouples into two evolution subproblems.

The Flexural Problem

$$\begin{cases} \mathrm{div}_\tau \mathrm{div}_\tau \widetilde{\mathbb{M}} - \frac{2h^3}{3}\rho\Delta_\tau \ddot{w} + 2h\rho\ddot{w} = \widetilde{f}_3 & \text{in } \omega \times (0,t_0), \\ w(0) = w_0,\ \dot{w}(0) = w_1 & \text{in } \omega, \\ \frac{2h^3}{3}\rho\nabla_\tau \ddot{w}\cdot\boldsymbol{\nu} - \mathrm{div}_\tau(\widetilde{\mathbb{M}}\boldsymbol{\nu}) - \nabla_\tau(\widetilde{\mathbb{M}}\boldsymbol{\nu}\cdot\boldsymbol{\tau})\cdot\boldsymbol{\tau} = \widetilde{g}_3 & \text{on } \gamma_1 \times (0,t_0), \\ \widetilde{\mathbb{M}}\boldsymbol{\nu}\cdot\boldsymbol{\nu} = 0 & \text{on } \gamma_1 \times (0,t_0), \\ w = \partial_\nu w = 0 & \text{on } \gamma_0 \times (0,t_0), \end{cases}$$

where $\widetilde{\mathbb{M}} := \dfrac{2h^3}{3}\widetilde{\mathbf{A}}\nabla_\tau\nabla_\tau w,\ \ \widetilde{\mathbf{A}} := \widetilde{\mathbf{C}} + \dfrac{1}{\widetilde{M_{33}}}\widetilde{\mathbf{R}}_3 \otimes \widetilde{\mathbf{R}}_3.$

Limit Evolution Problems

- The limit problem decouples into two evolution subproblems.

The 2D Thermo-Piezoelectric Evolution Problem

$$\begin{cases} 2h\rho\ddot{\mathbf{u}}_H - \mathrm{div}_\tau \widetilde{\mathbb{N}} = \widetilde{\mathbf{s}} + \widetilde{\mathbf{R}}_3[\![\nabla_\tau \zeta]\!] & \text{in } \omega \times (0, t_0), \\ \mathrm{div}_\tau \widetilde{\mathbf{D}} = \widetilde{d} + \widetilde{\boldsymbol{\alpha}}_3 \cdot [\![\nabla_\tau \zeta]\!] & \text{in } \omega \times (0, t_0), \\ \dot{\widetilde{\mathcal{S}}} + \mathrm{div}_\tau \widetilde{\mathbf{q}} = \widetilde{h} + \widetilde{m}_3 [\![\dot{\zeta}]\!] & \text{in } \omega \times (0, t_0), \\ \mathbf{u}_H(0) = \mathbf{u}_{H,0},\ \dot{\mathbf{u}}_H(0) = \mathbf{u}_{H,1},\ \vartheta(0) = \vartheta_0 & \text{in } \omega, \\ \widetilde{\mathbb{N}}\boldsymbol{\nu} = \widetilde{\mathbf{r}} - [\![\zeta]\!]\widetilde{\mathbf{R}}_3\boldsymbol{\nu} & \text{on } \gamma_1 \times (0, t_0), \\ \widetilde{\mathbf{D}} \cdot \boldsymbol{\nu} = [\![\zeta]\!]\widetilde{\boldsymbol{\alpha}}_3 \cdot \boldsymbol{\nu} & \text{on } \gamma_1 \times (0, t_0), \\ -\widetilde{\mathbf{q}} \cdot \boldsymbol{\nu} = \widetilde{\varrho} & \text{on } \gamma_1 \times (0, t_0), \\ \mathbf{u}_H = \mathbf{0}, \quad \phi = \vartheta = 0 & \text{on } \gamma_0 \times (0, t_0), \end{cases}$$

where

$$\begin{cases} \widetilde{\mathbb{N}} := 2h(\widetilde{\mathbf{C}}\, \widetilde{\mathbf{e}}(\mathbf{u}_H) + \widetilde{\mathbf{P}}\nabla_\tau \phi - \widetilde{\boldsymbol{\beta}}\vartheta), \\ \widetilde{\mathbf{D}} := 2h(\widetilde{\mathbf{P}}^T \widetilde{\mathbf{e}}(\mathbf{u}_H) - \widetilde{\mathbf{X}}\nabla_\tau \phi + \widetilde{\mathbf{p}}\vartheta), \\ \widetilde{\mathcal{S}} := 2h(\widetilde{\boldsymbol{\beta}} : \widetilde{\mathbf{e}}(\mathbf{u}_H) - \widetilde{\mathbf{p}} \cdot \nabla_\tau \phi + \widetilde{c}_v \vartheta), \\ \widetilde{\mathbf{q}} := -\frac{2h}{T_0}\widetilde{\mathbf{Q}}\nabla_\tau \vartheta. \end{cases}$$

References

- S. N. Ahmad, C. S. Upadhyay, C. Venkatesan, Electro-thermo-elastic formulation for the analysis of smart structures, J. Smart Materials and Structures, 15 No 2, (2006), 401-416.
- S. Imperiale, P. Joly, Mathematical and numerical modelling of piezoelectric sensors, ESAIM: Mathematical Modelling and Numerical Analysis (2012), 875-909.
- B. Miara, J. S. Suárez, Asymptotic pyroelectricity and pyroelasticity in thermopiezoelectric plates, Asymptotic Analysis 81 (2013), 211-250.
- P. Kondaiah, K. Shankar, N. Ganesan, Pyroelectric and pyromagnetic effects on behavior of magneto-electro-elastic plate, Coupled Systems Mechanics, Vol. 2, No. 1 (2013), 1-22.
- T. Weller, C. Licht, Asymptotic modeling of thin piezoelectric plates, Ann Solid Struct Mech 1, (2010), 173-188.
- T. Weller, C. Licht, Mathematical modeling of piezomagnetoelectric thin plates. Eur J Mech A-Solids 29, (2010) 928-937.
- P.G. Ciarlet, *Mathematical Elasticity, vol. II: Theory of Plates*, North-Holland, Amsterdam (1997).
- T. Weller, C. Licht, Modeling of linearly electromagneto-elastic thin plates, C. R. Mecanique 335, (2007), 201-206.

Material constants for a $BaTiO_3$-$CoFe_2O_4$ complex

Table 1 Material properties of *PZT-5* and magneto-electro-thermo-elastic composite with volume fraction, v_f = 0.6 of $BaTiO_3$ (Chen *et al.* (2007), Challagulla and Georgiades (2011), Aboudi (2001), Biju *et al.* (2011))

Elastic constants:	$v_f = 0.6$	*PZT-5*	Magnetic Permeability:	$v_f = 0.6$	*PZT-5*
$c_{11} = c_{22}$ (GPa)	200	99.2	$\mu_{11} = \mu_{22}$ (10^{-4}N s^2/C^2)	-1.5	-
c_{12} (GPa)	110	54.01	μ_{33} (10^{-4}N s^2/C^2)	0.75	-
$c_{13} = c_{23}$ (GPa)	110	50.77	Piezomagnetic constants:		
c_{33} (GPa)	190	86.85	$q_{31} = q_{32}$ (N/A m)	200	-
$c_{44} = c_{55}$ (GPa)	45	21.1	q_{33} (N/A m)	260	-
c_{66} (GPa)	45	22.593	q_{15} (N/A m)	180	-
Piezoelectric constants:			Magnetoelectric constant:		
$e_{31} = e_{32}$ (C/m^2)	-3.5	-7.20	$m_{11} = m_{22}$ (10^{-12} N s/V C)	6	-
e_{33} (C/m^2)	11	15.11	m_{33} (10^{-12} N s/V C)	2500	-
e_{15} (C/m^2)	0	12.32	Pyroelectric constants:		
Dielectric constant:			p_2 (10^{-5} C/m^2 K)	-12.4	
$\varepsilon_{11} = \varepsilon_{22}$ (10^{-9}C^2/N m^2)	0.9	1.53	Pyromagnetic constants:		
ε_{33} (10^{-9}C^2/N m^2)	7.5	1.5	τ_2 (10^{-3} N/A m K)	5.92	-
Thermal expansion coefficients:			Density:		
$\beta_{11} = \beta_{22}$ (10^{-6} 1/K)	12.9	1.5	ρ (kg/m^3)	5600	7750
β_{33} (10^{-6} 1/K)	7.8	2			