Fast and accurate floating-point division on ST231

Algorithm, implementation, and automatic generation and validation

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Context and objectives

Context

- FLIP software library
  - [http://flip.gforge.inria.fr/](http://flip.gforge.inria.fr/)
  - support for floating-point arithmetic on integer processors
- low latency implementation of binary floating-point division
  - targets a VLIW integer processor of the ST200 family
- no support of *subnormal* numbers
  - input/output: ±0, ±∞, NaN or *normal* number

Objectives

- faster software implementation (compared to FLIP 0.3)
  - expose instruction-level parallelism via bivariate polynomial evaluation
- correctly rounded
  - rounding-to-nearest even
Notation and assumptions

- **Input** \((x, y)\): two positive normal numbers
  \[ x = (-1)^{s_x} \cdot m_x \cdot 2^{e_x} \quad \text{with} \quad \begin{cases} s_x \in \{0, 1\} \\ m_x = 1.m_{x,1} \ldots m_{x,p-1} \in [1, 2) \\ e_x \in \{e_{\min}, \ldots, e_{\max}\} \end{cases} \]

- **Computation**: \(k\)-bit unsigned integers
  \[ \rightarrow \text{register size } k \]

- **Example for binary32 format**: \((k, p, e_{\max}) = (32, 24, 127)\)
Outline of the talk

Division via polynomial evaluation

Generation of an efficient evaluation program

Validation of the generated evaluation program

Experimental results

Concluding remarks
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Division algorithm flowchart

Definition

\[ c = \begin{cases} 
1 & \text{if } m_x \geq m_y, \\
0 & \text{if } m_x < m_y.
\end{cases} \]
Division algorithm flowchart

- **Definition**

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c = \begin{cases} 
1 & \text{if } m_x \geq m_y, \\
0 & \text{if } m_x < m_y. 
\end{cases}
\]

- **Range reduction**

\[
x/y = \left(2^{m_x/m_y} \cdot 2^{-c}\right) \times 2^{e_x-e_y-1+c}
\]

\[
\ell = 2^{m_x/m_y} \cdot 2^{-c}
\]

\[
\ell \in [1, 2)
\]

\[
\text{RN}_p(\ell) \in [1, 2)
\]

\[
\text{RN}_p(x/y) = \text{RN}_p(\ell) \times 2^d
\]
Division algorithm flowchart

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- **Range reduction**

  \[
  x/y = (2m_x/m_y \cdot 2^{-c}) \times 2^{e_x-e_y-1+c}
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How to compute a correctly rounded significand?

- Iterative methods (restoring, non-restoring, ...)
  - Oberman and Flynn (1997)
  - minimal instruction-level parallelism exposure, sequential algorithm
How to compute a correctly rounded significand?

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- **Multiplicative methods** (Newton-Raphson, Goldschmidt)
  - more instruction-level parallelism exposure
  - previous implementation of division (FLIP 0.3)
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- **Polynomial-based methods**
  - Agarwal, Gustavson and Schmookler (1999)
    → univariate polynomial evaluation
  - Our approach
    → **single bivariate polynomial evaluation**
Truncated one-sided approximation

- See for example, Ercegovac and Lang (2004)
- 3 steps
  1. compute \( v = (01.v_1 \ldots v_{k-2}) \) such that
     \[ -2^{-p} \leq \ell - v < 0 \]
     that is implied by \( |(\ell + 2^{-p-1}) - v| < 2^{-p-1} \)
  2. truncate \( v \) after \( p \) fraction bits
  3. obtain \( \text{RN}_p(\ell) \) after possibly adding \( 2^{-p} \)
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How to compute the one-sided approximation \( v \) ?
Computation of the one-sided approximation

1. Consider $\ell + 2^{-p-1}$ as the exact result of the function

\[ F(s, t) = s/(1 + t) + 2^{-p-1}, \]

at the points $s^* = 2^{1-c} m_x$ and $t^* = m_y - 1$:

\[ \ell + 2^{-p-1} = F(s^*, t^*). \]
Computing the one-sided approximation

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2. Approximate \( F(s, t) \) by a bivariate polynomial \( P(s, t) \)

\[
P(s, t) = s \cdot a(t) + 2^{-p-1}.
\]

\( a(t) \): univariate polynomial approximant of \( 1/(1 + t) \)

\( \rightarrow \) approximation entails an error \( \epsilon_{\text{approx}} \)
Computation of the one-sided approximation

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\[ v = \mathcal{P}(s^*, t^*). \]

→ evaluation entails an error $\epsilon_{\text{eval}}$
Computation of the one-sided approximation

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How to ensure that $|(\ell + 2^{-p-1}) - v| < 2^{-p-1}$?
Sufficient error bounds

Since by triangular inequality

$$|(\ell + 2^{-p-1}) - v| \leq \mu \cdot \epsilon_{\text{approx}} + \epsilon_{\text{eval}}$$

with

$$\mu = \max\{s^*\} = \max\{2^{1-c} m_x\} = \left(4 - 2^{3-p}\right)$$
Sufficient error bounds

- Since by triangular inequality
  \[ |(\ell + 2^{-p-1}) - v| \leq \mu \cdot \epsilon_{\text{approx}} + \epsilon_{\text{eval}} \]
  with
  \[ \mu = \max\{s^*\} = \max\{2^{1-c}m_x\} = (4 - 2^{3-p}) \]

- One has to ensure
  \[ \mu \cdot \epsilon_{\text{approx}} + \epsilon_{\text{eval}} < 2^{-p-1} \]

- Sufficient conditions can be obtained
  \[ \epsilon_{\text{approx}} < 2^{-(p-1)}/\mu \quad \text{and} \quad \epsilon_{\text{eval}} < 2^{-(p-1)} - \mu \cdot \epsilon_{\text{approx}} \]
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Automatic generation of an efficient evaluation program

- Evaluation program $\mathcal{P} =$ main part of the full software implementation
  - dominates the cost

- By **efficient**, one means an evaluation program that
  - reduces the evaluation latency
  - reduces the number of multiplications
  - is accurate enough
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- Target architecture: ST231
  - 4-issue VLIW integer processor with at most 2 mul. per cycle
  - latencies: addition = 1 cycle, multiplication = 3 cycles
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Which evaluation program to evaluate the polynomial \( P(s, t) \)?
Example for the binary32 implementation: \((k, p) = (32, 24)\)

\[
P(s, t) = 2^{-p-1} + s \cdot \sum_{i=0}^{10} a_i t^i
\]

- Horner’s scheme: \((3 + 1) \times 11 = 44\) cycles
  - sequential scheme, no instruction-level parallelism exposure
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- Estrin’s scheme: 20 cycles
  - more instruction-level parallelism
  - a last multiplication by \(s\)
  - 2 cycles save by distributing the multiplication by \(s\) in the evaluation of the univariate polynomial \(a(t)\)
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- ...

We can do much better.

But how to explore the solution space and choose an efficient evaluation program?
  - interest of automatic generation
Efficient evaluation tree generation

- Similar to Harrison, Kubaska, Story and Tang (1999)

- Assumption
  - unbounded parallelism
  - latencies of arithmetic operators: + and ×
Efficient evaluation tree generation

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- Assumption
  - unbounded parallelism
  - latencies of arithmetic operators: $+$ and $\times$

- Two sub-steps
  1. determine a target latency $\tau$
     
     ie. $\tau = 3 \times \lceil \log_2(\deg(P)) \rceil + 1$
  2. generate automatically a set of evaluation trees, with height $\leq \tau$
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⇒ if no tree satisfies $\tau$ then increase $\tau$ and restart
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  $\Rightarrow$ if no tree satisfies $\tau$ then increase $\tau$ and restart

- Number of evaluation trees = extremely large $\rightarrow$ several filters
Efficient evaluation tree generation

\[ P(s, t) = 2^{-p-1} + s \cdot \sum_{i=0}^{10} a_i t^i \]

- first target latency \( \tau = 13 \)
  \[ \rightarrow \text{no tree found} \]
 Efficient evaluation tree generation

\[ P(s, t) = 2^{-p-1} + s \cdot \sum_{i=0}^{10} a_i t^i \]

- first target latency \( \tau = 13 \)
  \( \rightarrow \) no tree found

- second target latency \( \tau = 14 \)
  \( \rightarrow \) obtained in about 10 sec.
Efficient evaluation tree generation

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- First target latency \( \tau = 13 \)
  - no tree found

- Second target latency \( \tau = 14 \)
  - obtained in about 10 sec.

- Distribute the multiplication by \( s \)
  - otherwise: 18 cycles

- Too difficult to find such tree by hand
Arithmetic operator choice

- Polynomial coefficients implemented in absolute value
- All intermediate values have constant sign
  - not store the sign: more accuracy
Arithmetic operator choice

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- Label evaluation trees by appropriate arithmetic operator: + or −
Arithmetic operator choice

- Polynomial coefficients implemented in absolute value
- All intermediate values have constant sign
  ⇒ not store the sign: more accuracy

- Label evaluation trees by appropriate arithmetic operator: + or –

- If the sign of an intermediate value changes when the input varies then the evaluation tree is rejected
  ⇒ implementation with certified interval arithmetic (MPFI)
Practical scheduling checking

- Schedule the evaluation trees on a simplified model of a real target architecture
  - operator costs, nb. issues, constraints on operators
  - no syllables constraint
Practical scheduling checking

- Schedule the evaluation trees on a *simplified model* of a real target architecture
  - operator costs, nb. issues, constraints on operators
  - no syllables constraint

- Check if no increase of latency in practice compared to the latency on unbounded parallelism
  - if practical latency > theoretical latency then the evaluation tree is rejected

  ⇒ implementation using *naive list scheduling algorithm* is enough
Outline of the talk

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- Validation of the generated evaluation program
- Experimental results
- Concluding remarks
Example for the binary32 implementation: \((k, p) = (32, 24)\)

- Approximation of \(1/(1 + t)\) by truncated Remez’ polynomial of degree 10

\[
\epsilon_{\text{approx}} \leq 2^{-27.41\ldots} \approx 6.0 \times 10^{-9} < 2^{-25}/(4 - 2^{-21}) \approx 7.4 \times 10^{-9}
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- Deduction of the evaluation error bound from \(\epsilon_{\text{approx}}\)

\[
\epsilon_{\text{eval}} < 2^{-25} - (4 - 2^{-21}) \cdot 2^{-27.41}\ldots \approx 2^{-26.9999}\ldots \approx 7.4 \cdot 10^{-9}.
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Example for the binary32 implementation: \((k, p) = (32, 24)\)

- **Case 1:** \(m_x \geq m_y \rightarrow\) condition satisfied
- **Case 2:** \(m_x < m_y \rightarrow\) condition not satisfied

ie. \(s^* = 3.935581684112548828125\) and \(t^* = 0.97490441799163818359375\)
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1. Determine an interval \(\mathcal{I}\) around this point
2. Compute \(\epsilon_{\text{approx}}\) over \(\mathcal{I}\)
3. Determine an evaluation error bound \(\eta\)
4. Check if \(\epsilon_{\text{eval}} < \eta\)?
Evaluation program validation strategy

- Find a splitting of the input interval into \( n \) subinterval(s) \( T^{(i)} \), and check that

\[
\mu \cdot \epsilon_{\text{approx}}^{(i)} + \epsilon_{\text{eval}}^{(i)} < 2^{-p-1}
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on each subinterval.
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- Implementation of the splitting by dichotomy

  - for each \( T^{(i)} \)
    1. compute a certified approximation error bound \( \epsilon_{\text{approx}}^{(i)} \)
    2. determine an evaluation error bound \( \epsilon_{\text{eval}}^{(i)} \)
    3. check this bound

\[\Rightarrow\] if this bound is not satisfied, \( T^{(i)} \) is split up into 2 subintervals

  - implemented using \textit{Sollya} (steps 1 and 2) and \textit{Gappa} (step 3)
Validation of the generated evaluation program

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  \( \Rightarrow \) if this bound is not satisfied, \( T^{(i)} \) is split up into 2 subintervals

  - implemented using Sollya (steps 1 and 2) and Gappa (step 3)

- Example of binary32 implementation

  - launched on a 64 processor grid
  - 36127 subintervals found in several hours (\( \approx 5h. \)
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Experimental results

Performances on ST231

<table>
<thead>
<tr>
<th></th>
<th>Nb. of instructions</th>
<th>Latency (# cycles)</th>
<th>IPC</th>
<th>Code size (bytes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>rounding to nearest</td>
<td>86</td>
<td>27</td>
<td>3.18</td>
<td>416</td>
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</tbody>
</table>

- speed-up by a factor of about 1.78 in rounding to nearest compared to the previous implementation (48 cycles)
  - optimized implementation
  - efficient ST200 compiler (st200cc)

- high IPC value: confirms the parallel nature of our approach
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Contributions

- New approach for the implementation of binary floating-point division
  - bivariate polynomial-based algorithm
  - automatic generation and validation of efficient evaluation program
  - implementation targeted ST231 VLIW integer processor

- Speed-up by a factor of about 1.78 in rounding to nearest compared to the previous implementation

Since then

- Extension to subnormal numbers support
  - implementation in 31 cycles: 4 extra cycles

- Implementation of other functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Latency (# cycles)</th>
<th>IPC</th>
<th>Code size (bytes)</th>
<th>Speed-up</th>
</tr>
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<tbody>
<tr>
<td>square root</td>
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<td>276</td>
<td>2.38</td>
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<td>reciprocal</td>
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<td>2.59</td>
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<td>1.75</td>
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<tr>
<td>reciprocal square root</td>
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<td>2.24</td>
<td>368</td>
<td>2.27</td>
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