Implementation of binary floating-point arithmetic on embedded integer processors

Polynomial evaluation-based algorithms and certified code generation

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**Motivation**

- **Embedded systems** are ubiquitous
  - microprocessors dedicated to one or a few specific tasks
  - satisfy constraints: area, energy consumption, conception cost
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Overview of the ST231 architecture

- 4-issue VLIW 32-bit integer processor  
  → no FPU
- Parallel execution unit  
  ▶ 4 integer ALU  
  ▶ 2 pipelined multipliers $32 \times 32 \rightarrow 32$
- Latencies: ALU → 1 cycle, Mul → 3 cycles
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  - instructions grouped into bundles
  - Instruction-Level Parallelism (ILP) explicitly exposed by the compiler
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```
uint32_t R1 = A0 + C;
uint32_t R2 = A3 * X;
uint32_t R3 = A1 * X;
uint32_t R4 = X * X;
```

<table>
<thead>
<tr>
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<th>Issue 1</th>
<th>Issue 2</th>
<th>Issue 3</th>
<th>Issue 4</th>
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</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>R4</td>
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</table>
How to emulate floating-point arithmetic in software?

Design and implementation of efficient software support for IEEE 754 floating-point arithmetic on integer processors

- Existing software for IEEE 754 floating-point arithmetic:
  - Software floating-point support of GCC, Glibc and µClibc, GoFast Floating-Point Library
  - SoftFloat (→ STlib)
  - FLIP (Floating-point Library for Integer Processors)
    - software support for binary32 floating-point arithmetic on integer processors
    - correctly-rounded addition, subtraction, multiplication, division, square root, reciprocal, ...
    - handling subnormals, and handling special inputs
Towards the generation of fast and certified codes

- **Underlying problem**: development “by hand”
  - long and tedious, error prone
  - new target ? new floating-point format ?
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■ Underlying problem: development “by hand”
  ▶ long and tedious, error prone
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  ⇒ need for automation and certification

■ Current challenge: tools and methodologies for the automatic generation of efficient and certified programs
  ▶ optimized for a given format, for the target architecture
Towards the generation of fast and certified codes

- Arénaire’s developments: hardware (FloPoCo) and software (Sollya, Metalibm)

- Spiral project: hardware and software code generation for DSP algorithms

  *Can we teach computers to write fast libraries?*
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  Can we teach computers to write fast libraries?

- Our tool: CGPE (Code Generation for Polynomial Evaluation)

  In the particular case of polynomial evaluation, can we teach computers to write fast and certified codes, for a given target and optimized for a given format?
Basic blocks for implementing correctly-rounded operators

(X, Y)

Special input detection

Floating-point number unpacking

Normalization

Range reduction

Result sign/exponent computation

Result significand approximation

Rounding condition decision

Correct rounding computation

Result reconstruction

Special output selection

function independent

function dependent

Objectives

→ Low latency, correctly-rounded implementations

→ ILP exposure
Basic blocks for implementing correctly-rounded operators

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- Problem: function to be evaluated
- Computation of polynomial approximant
- Efficient and certified C code generation
- C code
- Certificate

ST231 features

- Fully automated

- Uniform approach for $n$th roots and their reciprocals
  \[ \rightarrow \text{polynomial evaluation} \]
- Extension to division

Flowchart for generating efficient and certified C codes

Problem: function to be evaluated

Computations of polynomial approximant

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ST231 features

Constraints
- Accuracy of approximant and C code
- Sollya
  - interval arithmetic (MPFI), Gappa
- Low evaluation latency on ST231, ILP exposure

Efficiency of the generation process
Flowchart for generating efficient and certified C codes

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- Efficiency of the generation process

CGPE
Outline of the talk

1. Design and implementation of floating-point operators
   Bivariate polynomial evaluation-based approach
   Implementation of correct rounding

2. Low latency parenthesization computation
   Classical evaluation methods
   Computation of all parenthesizations
   Towards low evaluation latency

3. Selection of effective evaluation parenthesizations
   General framework
   Automatic certification of generated C codes

4. Numerical results

5. Conclusions and perspectives
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1. Design and implementation of floating-point operators
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Notation and assumptions

Input $(x, y)$ and output $\text{RN}(x/y)$: normal numbers

- no underflow nor overflow
- precision $p$, extremal exponents $e_{\text{min}}, e_{\text{max}}$

\[ x = \pm 1.m_{x,1} \ldots m_{x,p-1} \cdot 2^{e_x} \quad \text{with} \quad e_x \in \{ e_{\text{min}}, \ldots, e_{\text{max}} \} \]
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- RoundTiesToEven
Notation and assumptions

- **Standard binary encoding**: $k$-bit unsigned integer $X$ encodes input $x$
  
  $s_x$ $E_x = e_x - e_{\min} - 1$ $T_x = m_{x,1} \cdots m_{x,p-1}$

  1 bit $w = k - p$ bits $p - 1$ bits

- **Computation**: $k$-bit unsigned integers
  
  $\rightarrow$ integer and fixed-point arithmetic
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  s_x \quad E_x = e_x - e_{\min} - 1 \quad T_x = m_{x,1} \ldots m_{x,p-1}
  \]

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  w = k - p \text{ bits} \quad p - 1 \text{ bits}
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- **Computation**: $k$-bit unsigned integers
  \[\rightarrow\] integer and fixed-point arithmetic
Range reduction of division

Express the exact result $r = x/y$ as:

$$\ell \in [1, 2) \quad \text{and} \quad d \in \{ e_{\min}, \ldots, e_{\max} \}$$
Range reduction of division

Express the exact result $r = x/y$ as:

$$r = \ell \cdot 2^d \Rightarrow \text{RN}(x/y) = \text{RN}(\ell) \cdot 2^d$$

with

$$\ell \in [1, 2) \quad \text{and} \quad d \in \{e_{\text{min}}, \ldots, e_{\text{max}}\}$$

Definition

$$c = 1 \quad \text{if} \quad m_x \geq m_y, \quad \text{and} \quad c = 0 \quad \text{otherwise}$$
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- Range reduction

$$x/y = (2^{1-c} \cdot m_x/m_y) \cdot 2^d \quad \text{with} \quad d = e_x - e_y - 1 + c$$

$$:= \ell \in [1,2)$$
Range reduction of division

- Express the exact result \( r = x / y \) as:
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  r = \ell \cdot 2^d \quad \Rightarrow \quad \text{RN}(x/y) = \text{RN}(\ell) \cdot 2^d
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  \]
  \[\boxed{:= \ell \in [1,2)}\]

How to compute the correctly-rounded significand \( \text{RN}(\ell) \)?
Methods for computing the correctly-rounded significand

- **Iterative methods**: restoring, non-restoring, SRT, ...
  - Oberman and Flynn (1997)
  - minimal ILP exposure, sequential algorithm
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- **Multiplicative methods**: Newton-Raphson, Goldschmidt
  - Piñeiro and Bruguera (2002) – Raina’s Ph.D., FLIP 0.3 (2006)
  - exploit available multipliers, more ILP exposure
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- **Polynomial-based methods**
  - Agarwal, Gustavson and Schmookler (1999)
    → univariate polynomial evaluation
  - Our approach
    → **bivariate polynomial evaluation**: maximal ILP exposure
Correct rounding via truncated one-sided approximation

- How to compute $\text{RN}(\ell)$, with $\ell = 2^{1-c} \cdot m_x / m_y$?

- **Three steps** for correct rounding computation

  1. compute $v = 1.v_1 \ldots v_{k-2}$ such that $-2^{-p} \leq \ell - v < 0$

     $\rightarrow$ implied by $|(\ell + 2^{-p-1}) - v| < 2^{-p-1}$

     $\rightarrow$ bivariate polynomial evaluation

  2. compute $u$ as the truncation of $v$ after $p$ fraction bits

  3. determine $\text{RN}(\ell)$ after possibly adding $2^{-p}$
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How to compute the one-sided approximation $v$ and then deduce $\text{RN}(\ell)$?
One-sided approximation via bivariate polynomials

1. Consider $\ell + 2^{-p-1}$ as the exact result of the function

$$F(s, t) = \frac{s}{1 + t} + 2^{-p-1}$$

at the points $s^* = 2^{1-c} \cdot m_x$ and $t^* = m_y - 1$
One-sided approximation via bivariate polynomials

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2. Approximate $F(s, t)$ by a bivariate polynomial $P(s, t)$

   $$P(s, t) = s \cdot a(t) + 2^{-p-1}$$

   $\rightarrow a(t)$: univariate polynomial approximant of $1/(1 + t)$

   $\rightarrow$ approximation error $E_{\text{approx}}$
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3. Evaluate $P(s, t)$ by a well-chosen efficient evaluation program $\mathcal{P}$

$$v = \mathcal{P}(s^*, t^*)$$

$\rightarrow$ evaluation error $E_{\text{eval}}$
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How to ensure that $|(\ell + 2^{-p-1}) - v| < 2^{-p-1}$?
Sufficient error bounds

To ensure \( |(\ell + 2^{-p-1}) - v| < 2^{-p-1} \)

it suffices to ensure that \( \mu \cdot E_{\text{approx}} + E_{\text{eval}} < 2^{-p-1} \),

since

\[
| (\ell + 2^{-p-1}) - v | \leq \mu \cdot E_{\text{approx}} + E_{\text{eval}} \quad \text{with} \quad \mu = 4 - 2^{3-p}
\]
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\(|(\ell + 2^{-p-1}) - v| \leq \mu \cdot E_{\text{approx}} + E_{\text{eval}}\) with \(\mu = 4 - 2^{3-p}\)

This gives the following sufficient conditions

\[E_{\text{approx}} < \frac{2^{-p-1}}{\mu} \quad \Rightarrow \quad E_{\text{eval}} < 2^{-p-1} - \mu \cdot E_{\text{approx}}\]
Sufficient error bounds

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This gives the following sufficient conditions

\[E_{\text{approx}} \leq \theta \text{ with } \theta < 2^{-p-1}/\mu \implies E_{\text{eval}} < \eta = 2^{-p-1} - \mu \cdot \theta\]
Example for the \textit{binary32} division

- Sufficient conditions with $\mu = 4 - 2^{-21}$

\[
E_{\text{approx}} \leq \theta \quad \text{with} \quad \theta < 2^{-25}/\mu \quad \text{and} \quad E_{\text{eval}} < \eta = 2^{-25} - \mu \cdot \theta
\]

\[\text{Approximation error} \]
\[\text{Required bound} \]
\[\text{Absolute approximation error} \]
\[\text{Approximation error} \]
\[\text{Required bound} \]
Example for the *binary32* division

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- Approximation of $1/(1 + t)$ by a *Remez-like polynomial* of degree 10

  - $E_{\text{approx}} \leq \theta,$
    
    with $\theta = 3 \cdot 2^{-29} \approx 6 \cdot 10^{-9}$
  
  - $E_{\text{eval}} < \eta,$
    
    with $\eta \approx 7.4 \cdot 10^{-9}$
Flowchart for generating efficient and certified C codes

\[ F(s,t) \quad E_{\text{approx}} \leq \theta \quad E_{\text{eval}} < \eta \]

- Computation of polynomial approximant
- Computation of low latency parenthesizations
- Selection of effective parenthesizations

\[ |(\ell + 2^{-p-1}) - v| < 2^{-p-1} \]
Rounding condition: definition

- Approximation $u$ of $\ell$ with

$$\ell = 2^{1-c} \cdot m_x/m_y$$

- The exact value $\ell$ may have an infinite number of bits
  $\rightarrow$ the sticky bit cannot always be computed
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**Rounding condition**: $u \geq \ell$

$$u \geq \ell \iff u \cdot m_y \geq 2^{1-c} \cdot m_x$$
Rounding condition: implementation in integer arithmetic

- Rounding condition: $u \cdot m_y \geq 2^{1-c} \cdot m_x$

- Approximation $u$ and $m_y$: representable with 32 bits

$u \cdot m_y$ is exactly representable with 64 bits
Rounding condition: implementation in integer arithmetic

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\[
\begin{array}{c}
\times \\
\hline
u \\
\downarrow \\
\hline
m_y \\
\downarrow \\
\hline \\
u \cdot m_y \\
\hline 2^{1-c} \cdot m_x
\end{array}
\]

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- \( 2^{1-c} \cdot m_x \) is representable with 32 bits since \( c \in \{0, 1\} \)

\[ u \cdot m_y \geq 2^{1-c} \cdot m_x \]

\[ \Rightarrow \text{one } 32 \times 32 \rightarrow 32\text{-bit multiplication and one comparison} \]
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C code  Certificate

ST231 features
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ST231 features
Outline of the talk

1. Design and implementation of floating-point operators

2. Low latency parenthesization computation
   - Classical evaluation methods
   - Computation of all parenthesizations
   - Towards low evaluation latency

3. Selection of effective evaluation parenthesizations

4. Numerical results

5. Conclusions and perspectives
Objectives

- Compute an efficient parenthesization for evaluating $P(s, t)$
  - reduces the evaluation latency on unbounded parallelism
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- Evaluation program $P = \text{main part of the full software implementation}$
  - dominates the cost
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- Two families of algorithms
  - algorithms with coefficient adaptation: Knuth and Eve (60’s), Paterson and Stockmeyer (1964), ...
    - ill-suited in the context of fixed-point arithmetic
  - algorithms without coefficient adaptation
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  - algorithms without coefficient adaptation
Classical parenthesizations for binary32 division

\[ P(s, t) = 2^{-25} + s \cdot \sum_{0 \leq i \leq 10} a_i t^i \]

- Horner's rule: \((3 + 1) \times 11 = 44\) cycles
  \[\rightarrow\] no ILP exposure
Classical parenthesesizations for *binary32* division

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- Horner’s rule: \((3 + 1) \times 11 = 44\) cycles
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- Second-order Horner’s rule: 27 cycles
  \[\rightarrow\] evaluation of odd and even parts independently with Horner, more ILP
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- **Estrin’s method:** 19 cycles  
  → evaluation of high and low parts in parallel, even more ILP  
  → distributing the multiplication by \(s\) in the evaluation of \(a(t)\) → 16 cycles
Classical parenthesizations for $binary32$ division

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  $\rightarrow$ distributing the multiplication by $s$ in the evaluation of $a(t)$ $\rightarrow$ 16 cycles

$\ldots$ We can do better.

How to explore the solution space of parenthesizations?
Algorithm for computing all parenthesizations

\[ a(x, y) = \sum_{0 \leq i \leq dx} \sum_{0 \leq j \leq ny} a_{i,j} \cdot x^i \cdot y^j \quad \text{with} \quad n = n_x + n_y, \quad \text{and} \quad a_{n_x, n_y} \neq 0 \]

Example

Let \( a(x, y) = a_{0,0} + a_{1,0} \cdot x + a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y \). Then

\[ a_{1,0} + a_{1,1} \cdot y \] is a valid expression, while \[ a_{1,0} \cdot x + a_{1,1} \cdot x \] is not.
Algorithm for computing all parenthesizations

\[ a(x, y) = \sum_{0 \leq i \leq dx} \sum_{0 \leq j \leq n_y} a_{i,j} \cdot x^i \cdot y^j \quad \text{with} \quad n = n_x + n_y, \quad \text{and} \quad a_{n_x, n_y} \neq 0 \]

Example

Let \( a(x, y) = a_{0,0} + a_{1,0} \cdot x + a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y \). Then

\[ a_{1,0} + a_{1,1} \cdot y \quad \text{is a valid expression,} \quad \text{while} \quad a_{1,0} \cdot x + a_{1,1} \cdot x \quad \text{is not.} \]

- Exhaustive algorithm: iterative process
  \[ \rightarrow \text{step} \ k = \text{computation of all the valid expressions of total degree} \ k \]

- 3 building rules for computing all parenthesizations
Rules for building valid expressions

Consider step $k$ of the algorithm

- $E^{(k)}$: valid expressions of total degree $k$
- $P^{(k)}$: powers $x^i y^j$ of total degree $k = i + j$
Rules for building *valid* expressions

Consider step $k$ of the algorithm

- $E^{(k)}$: valid expressions of total degree $k$
- $P^{(k)}$: powers $x^i y^j$ of total degree $k = i + j$

Rule R1 for building the powers

\[
\text{deg}(p) = \text{deg}(p_1) + \text{deg}(p_2)
\]

\[
\deg(p_1) \leq \lceil k/2 \rceil \leq \deg(p_2) < k
\]
Rules for building valid expressions

Consider step $k$ of the algorithm

- $E^{(k)}$: valid expressions of total degree $k$
- $P^{(k)}$: powers $x^i y^j$ of total degree $k = i + j$

Rule R2 for expressions by multiplications

\[
\text{deg}(e) = \text{deg}(e') + \text{deg}(p)
\]

\[
\begin{align*}
\text{deg}(e') &< k \\
\text{deg}(p) &\leq k
\end{align*}
\]
Rules for building *valid* expressions

Consider step $k$ of the algorithm

- $E^{(k)}$: valid expressions of total degree $k$
- $P^{(k)}$: powers $x^i y^j$ of total degree $k = i + j$

Rule R3 for expressions by additions

\[
\text{deg}(e) = \max(\text{deg}(e_1), \text{deg}(e_2))
\]

\[
\begin{align*}
\text{deg}(e_1) &= k \\
\text{deg}(e_2) &\leq k
\end{align*}
\]
Number of parenthesizations

<table>
<thead>
<tr>
<th></th>
<th>$n_x = 1$</th>
<th>$n_x = 2$</th>
<th>$n_x = 3$</th>
<th>$n_x = 4$</th>
<th>$n_x = 5$</th>
<th>$n_x = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_y = 0$</td>
<td>1</td>
<td>7</td>
<td>163</td>
<td>11602</td>
<td>2334244</td>
<td>1304066578</td>
</tr>
<tr>
<td>$n_y = 1$</td>
<td>51</td>
<td>67467</td>
<td>1133220387</td>
<td>207905478247998</td>
<td>···</td>
<td>···</td>
</tr>
<tr>
<td>$n_y = 2$</td>
<td>67467</td>
<td>106191222651</td>
<td>10139277122276921118</td>
<td>···</td>
<td>···</td>
<td>···</td>
</tr>
</tbody>
</table>

Number of generated parenthesizations for evaluating a bivariate polynomial

- **Timings for parenthesization computation**
  - for univariate polynomial of degree $5 \approx 1h$ on a 2.4 GHz core
  - for bivariate polynomial of degree $(2,1) \approx 30s$
  - for $P(s, t)$ of degree $(3,1) \approx 7s$ (88384 schemes)

- **Optimization for univariate polynomial and $P(s, t)$**
  - univariate polynomial of degree $5 \approx 4min$
  - for $P(s, t)$ of degree $(3,1) \approx 2s$ (88384 schemes)
Number of parenthesizations

→ minimal latency for univariate polynomial of degree 5: 10 cycles (36 schemes)
Number of parenthesizations

<table>
<thead>
<tr>
<th>Number of degree-5 parenthesizations</th>
<th>Latency on unbounded parallelism (# cycles)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>1000</td>
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<tr>
<td></td>
<td>100000</td>
</tr>
<tr>
<td></td>
<td>1e+06</td>
</tr>
</tbody>
</table>

→ minimal latency for univariate polynomial of degree 5: 10 cycles (36 schemes)

How to compute only parenthesizations of low latency?
Determination of a *target* latency

- Target latency = *minimal cost* for evaluating

\[ a_{0,0} + a_{nx,ny} \cdot x^{nx} y^{ny} \]

- if no scheme satisfies \( \tau \) then increase \( \tau \) and restart
Determination of a **target latency**

- Target latency = **minimal cost** for evaluating

\[ a_{0,0} + a_{n_x, n_y} \cdot x^{n_x} y^{n_y} \]

- if no scheme satisfies \( \tau \) then increase \( \tau \) and restart

- **Static target latency** \( \tau_{\text{static}} \)
  - as general as evaluating \( a_{0,0} + x^{n_x+n_y+1} \)

\[
\tau_{\text{static}} = A + M \times \left\lceil \log_2(n_x + n_y + 1) \right\rceil
\]
Determination of a \textit{target} latency

- Target latency = \textit{minimal cost} for evaluating

\[ a_{0,0} + a_{n_x,n_y} \cdot x^{n_x} y^{n_y} \]

- if no scheme satisfies \( \tau \) then increase \( \tau \) and restart

- Static target latency \( \tau_{\text{static}} \)
  - as general as evaluating \( a_{0,0} + x^{n_x+n_y+1} \)
  \[ \tau_{\text{static}} = A + M \times \lceil \log_2(n_x + n_y + 1) \rceil \]

- Dynamic target latency \( \tau_{\text{dynamic}} \)
  - cost of operator on \( a_{n_x,n_y} \) and delay on interdeterminates
  - dynamic programming
Determination of a \textit{target} latency

- Target latency = \textit{minimal cost} for evaluating

\[ a_{0,0} + a_{n_x,n_y} \cdot x^{n_x} y^{n_y} \]

- if no scheme satisfies $\tau$ then increase $\tau$ and restart

Example

- Degree-9 bivariate polynomial: $n_x = 8$ and $n_y = 1$
- Latencies: $A = 1$ and $M = 3$
- Delay: $y$ available 9 cycles later than $x$

<table>
<thead>
<tr>
<th>$\tau_{\text{static}}$</th>
<th>$\tau_{\text{dynamic}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 + 3 \times \lceil \log_2(10) \rceil = 13$ cycles</td>
<td>16 cycles</td>
</tr>
</tbody>
</table>
Optimized search of *best* parenthesizations

Example

Let $a(x, y)$ be a degree-2 bivariate polynomial

$$a(x, y) = a_{0,0} + a_{1,0} \cdot x + a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y.$$ 

⇒ find a best *splitting* of the polynomial → low latency
Optimized search of *best* parenthesizations

Example

Let \( a(x, y) \) be a degree-2 bivariate polynomial

\[
a(x, y) = a_{0,0} + a_{1,0} \cdot x + a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y.
\]

⇒ find a best **splitting** of the polynomial → low latency

\[
\left( a_{0,0} + a_{1,0} \cdot x + a_{0,1} \cdot y \right) + \left( a_{1,1} \cdot x \cdot y \right)
\]
Optimized search of *best* parenthesizations

Example

Let $a(x, y)$ be a degree-2 bivariate polynomial

$$a(x, y) = a_{0,0} + a_{1,0} \cdot x + a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y.$$ 

⇒ find a best splitting of the polynomial → low latency

$$\left( \left( a_{0,0} + a_{1,0} \cdot x \right) + a_{0,1} \cdot y \right) + \left( a_{1,1} \cdot x \cdot y \right)$$
Optimized search of *best* parenthesizations

Example
Let \( a(x, y) \) be a degree-2 bivariate polynomial

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a(x, y) = a_{0,0} + a_{1,0} \cdot x + a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y.
\]

⇒ find a best *splitting* of the polynomial → low latency

\[
\left( a_{0,0} + (a_{1,0} \cdot x + a_{0,1} \cdot y) \right) + \left( a_{1,1} \cdot x \cdot y \right)
\]
Optimized search of best parenthesesizations

Example

Let \( a(x, y) \) be a degree-2 bivariate polynomial

\[
a(x, y) = a_{0,0} + a_{1,0} \cdot x + a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y.
\]

\(\Rightarrow\) find a best splitting of the polynomial \(\rightarrow\) low latency

\[
\left( a_{0,0} + a_{1,0} \cdot x \right) + \left( a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y \right)
\]
Optimized search of best parenthesizations

Example

Let $a(x, y)$ be a degree-2 bivariate polynomial

$$a(x, y) = a_{0,0} + a_{1,0} \cdot x + a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y.$$  

⇒ find a best splitting of the polynomial → low latency

$$a_{0,0} + \left( a_{1,0} \cdot x + a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y \right)$$
Optimized search of *best* parenthesizations

Example

Let $a(x, y)$ be a degree-2 bivariate polynomial

$$a(x, y) = a_{0,0} + a_{1,0} \cdot x + a_{0,1} \cdot y + a_{1,1} \cdot x \cdot y.$$  

⇒ find a best splitting of the polynomial → low latency
Efficient evaluation parenthesization generation

\[ P(s, t) = 2^{-25} + s \cdot \sum_{0 \leq i \leq 10} a_i t^i \]

- First target latency \( \tau = 13 \)
  - \( \rightarrow \) no parenthesization found
Efficient evaluation parenthesization generation

\[ P(s, t) = 2^{-25} + s \cdot \sum_{0 \leq i \leq 10} a_i t^i \]

- First target latency \( \tau = 13 \)
  \[ \rightarrow \text{no parenthesization found} \]

- Second target latency \( \tau = 14 \)
  \[ \rightarrow \text{obtained in about 10 sec.} \]

- Classical methods
  - Horner: 44 cycles,
  - Estrin: 19 cycles,
  - Estrin by distributing \( s \): 16 cycles
Flowchart for generating efficient and certified C codes

\[
F(s,t) \quad E_{\text{approx}} \leq \theta \quad E_{\text{eval}} < \eta
\]

Computation of polynomial approximant

\[a(t)\]

Computation of low latency parenthesizations

Selection of effective parenthesizations

??

C code

Certificate

ST231 features
Flowchart for generating efficient and certified C codes

\[ F(s,t) \quad E_{\text{approx}} \leq \theta \quad E_{\text{eval}} < \eta \]

- Computation of polynomial approximant
- Selection of effective parenthesizations
- C code
- Certificate

ST231 features
Outline of the talk

1. Design and implementation of floating-point operators

2. Low latency parenthesization computation

3. Selection of effective evaluation parenthesizations
   - General framework
   - Automatic certification of generated C codes

4. Numerical results

5. Conclusions and perspectives
Selection of effective evaluation parenthesizations

1. Arithmetic Operator Choice
   - all intermediate variables are of constant sign

2. Scheduling on a simplified model of the ST231
   - constraints of architecture: cost of operators, instructions bundling, ...
   - delays on indeterminates

3. Certification of generated C code
   - straightline polynomial evaluation program
   - "certified C code": we can bound the evaluation error in integer arithmetic
Selection of effective parenthesizations

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3. Certification of generated C code
   ▶ straightline polynomial evaluation program
   ▶ “certified C code”: we can bound the evaluation error in integer arithmetic
Certification of evaluation error for binary32 division

- Sufficient conditions with $\mu = 4 - 2^{-21}$

$$E_{\text{approx}} \leq \theta \quad \text{with} \quad \theta < 2^{-25}/\mu \quad \text{and} \quad E_{\text{eval}} < \eta = 2^{-25} - \mu \cdot \theta$$

- $E_{\text{approx}} \leq \theta$,

  with $\theta = 3 \cdot 2^{-29} \approx 6 \cdot 10^{-9}$

- $E_{\text{eval}} < \eta$,

  with $\eta \approx 7.4 \cdot 10^{-9}$
Certification of evaluation error for binary32 division

- **Case 1:** $m_x \geq m_y \rightarrow$ condition satisfied
- **Case 2:** $m_x < m_y \rightarrow$ condition not satisfied: $E_{\text{eval}} \geq \eta$

$s^* = 3.935581684112548828125$ and $t^* = 0.97490441799163818359375$

![Graph showing absolute approximation error vs. t](image-url)
Certification of evaluation error for *binary32* division

- **Case 1:** $m_x \geq m_y \rightarrow$ condition satisfied
- **Case 2:** $m_x < m_y \rightarrow$ condition not satisfied: $E_{\text{eval}} \geq \eta$

$s^* = 3.935581684112548828125$ and $t^* = 0.97490441799163818359375$

1. determine an interval $I$ around this point

Approximation error bounds:
- Required bound $2^{-25}/(4 - 2^{-21}) \approx 8 \cdot 10^{-9}$
- Approximation error bound $\theta = 3 \cdot 2^{-29} \approx 6 \cdot 10^{-9}$
Certification of evaluation error for *binary32* division

- **Case 1**: $m_x \geq m_y$ → condition satisfied
- **Case 2**: $m_x < m_y$ → condition not satisfied: $E_{\text{eval}} \geq \eta$

$s^* = 3.935581684112548828125$ and $t^* = 0.97490441799163818359375$

1. determine an interval $I$ around this point
2. compute $E_{\text{approx}}$ over $I$
3. determine an evaluation error bound $\eta$
4. check if $E_{\text{eval}} < \eta$?

---

**Approximation error**

Required bound $2^{-25}/(4 - 2^{-21}) \approx 8 \cdot 10^{-9}$

Approximation error bound $\theta = 3 \cdot 2^{-29} \approx 6 \cdot 10^{-9}$
Certification of evaluation error for \textit{binary32} division

- Sufficient conditions \textit{for each subinterval}, with $\mu = 4 - 2^{-21}$

\[
E_{\text{approx}}^{(i)} \leq \theta^{(i)} \quad \text{with} \quad \theta^{(i)} < 2^{-25}/\mu \quad \text{and} \quad E_{\text{eval}}^{(i)} < \eta^{(i)} = 2^{-25} - \mu \cdot \theta^{(i)}
\]
Certification of evaluation error for *binary32* division

- Sufficient conditions for each subinterval, with $\mu = 4 - 2^{-21}$

\[
E^{(i)}_{\text{approx}} \leq \theta^{(i)} \quad \text{with} \quad \theta^{(i)} < 2^{-25}/\mu \quad \text{and} \quad E^{(i)}_{\text{eval}} < \eta^{(i)} = 2^{-25} - \mu \cdot \theta^{(i)}
\]
Certification using a dichotomy-based strategy

- Implementation of the splitting by dichotomy

  - for each $\mathcal{T}^{(i)}$
    1. compute a certified approximation error bound $\theta^{(i)}$
    2. determine an evaluation error bound $\eta^{(i)}$
    3. check this bound: $E_{\text{eval}}^{(i)} < \eta^{(i)}$

  $\Rightarrow$ if this bound is not satisfied, $\mathcal{T}^{(i)}$ is split up into 2 subintervals
Certification using a dichotomy-based strategy

- Implementation of the splitting by dichotomy

  - for each $\mathcal{T}^{(i)}$
    1. compute a certified approximation error bound $\theta^{(i)}$
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Certification using a dichotomy-based strategy

- Implementation of the splitting by dichotomy

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    1. compute a certified approximation error bound $\theta^{(i)}$
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    3. check this bound: $E_{\text{eval}}^{(i)} < \eta^{(i)}$

      ⇒ if this bound is not satisfied, $\mathcal{T}^{(i)}$ is split up into 2 subintervals

- Example of binary32 implementation
  → launched on a 64 processor grid
  → 36127 subintervals found in several hours ($\approx 5h.$)
Outline of the talk

1. Design and implementation of floating-point operators
2. Low latency parenthesization computation
3. Selection of effective evaluation parenthesizations
4. Numerical results
5. Conclusions and perspectives
Performances of FLIP on ST231

⇒ Speed-up between 20 and 50 %
Performances of FLIP on ST231

Performances on ST231, in RoundTiesToEven

⇒ Speed-up between 20 and 50 %

Implementations of other operators

<table>
<thead>
<tr>
<th>Operator</th>
<th>$x^{-1}$</th>
<th>$x^{-1/2}$</th>
<th>$x^{1/3}$</th>
<th>$x^{-1/3}$</th>
<th>$x^{-1/4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>25</td>
<td>29</td>
<td>34</td>
<td>40</td>
<td>42</td>
</tr>
</tbody>
</table>

Performances on ST231, in RoundTiesToEven (in number of cycles)
## Impact of dynamic target latency

<table>
<thead>
<tr>
<th></th>
<th>$x^{1/3}$</th>
<th>$x^{-1/3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Degree</strong> $(n_x, n_y)$</td>
<td>(8,1)</td>
<td>(9,1)</td>
</tr>
<tr>
<td><strong>Delay on the operand $s$ (# cycles)</strong></td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td><strong>Static target latency</strong></td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td><strong>Dynamic target latency</strong></td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td><strong>Latency on unbounded parallelism and on ST231</strong></td>
<td>16</td>
<td>16</td>
</tr>
</tbody>
</table>

Latency (# cycles) on unbounded parallelism and on ST231.
## Impact of dynamic target latency

<table>
<thead>
<tr>
<th>Degree ((n_x, n_y))</th>
<th>Degree (x^{1/3})</th>
<th>Degree (x^{-1/3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delay on the operand (s) ((#) cycles)</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>Static <em>target</em> latency</td>
<td>13</td>
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<td>16</td>
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<td>16</td>
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</tr>
</tbody>
</table>

Latency \((\#\) cycles\) on unbounded parallelism and on ST231

\[ \Rightarrow \text{Conclude on the optimality in terms of polynomial evaluation latency} \]
### Timings for code generation

<table>
<thead>
<tr>
<th></th>
<th>$x^{1/2}$</th>
<th>$x^{-1/2}$</th>
<th>$x^{1/3}$</th>
<th>$x^{-1/3}$</th>
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</tr>
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<tbody>
<tr>
<td><strong>Degree ($n_x,n_y$)</strong></td>
<td>(8,1)</td>
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<td>(9,1)</td>
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</tr>
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<td><strong>Static target latency</strong></td>
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</tr>
<tr>
<td><strong>Latency on ST231</strong></td>
<td>13</td>
<td>14</td>
<td>16</td>
<td>16</td>
<td>13</td>
</tr>
</tbody>
</table>

| **Parenthesization generation** | 172ms | 152ms | 53s | 56s | 168ms |
| **Arithmetic Operator Choice**   | 6ms   | 6ms   | 7ms | 11ms | 4ms   |
| **Scheduling**                  | 29s   | 4m21s | 32ms | 132ms | 7s   |
| **Certification (Gappa)**        | 6s    | 4s    | 1m38s | 1m07s | 11s   |
| **Total time ($\approx$)**      | 35s   | 4m25s | 2m31s | 2m03s | 18s   |

**Timing of each step of the generation flow**
## Timings for code generation

<table>
<thead>
<tr>
<th></th>
<th>$x^{1/2}$</th>
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</tbody>
</table>

### Timing of each step of the generation flow

- **Impact of the target latency on the first step of the generation**
## Timings for code generation

<table>
<thead>
<tr>
<th>Degree ((n_x, n_y))</th>
<th>(x^{1/2})</th>
<th>(x^{-1/2})</th>
<th>(x^{1/3})</th>
<th>(x^{-1/3})</th>
<th>(x^{-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8,1)</td>
<td>(9,1)</td>
<td>(8,1)</td>
<td>(9,1)</td>
<td>(10,0)</td>
<td></td>
</tr>
<tr>
<td>Static target latency</td>
<td>13</td>
<td>13</td>
<td>13</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>Dynamic target latency</td>
<td>13</td>
<td>13</td>
<td>16</td>
<td>16</td>
<td>13</td>
</tr>
<tr>
<td>Latency on unbounded parallelism</td>
<td>13</td>
<td>13</td>
<td>16</td>
<td>16</td>
<td>13</td>
</tr>
<tr>
<td>Latency on ST231</td>
<td>13</td>
<td>14</td>
<td>16</td>
<td>16</td>
<td>13</td>
</tr>
</tbody>
</table>

| Parenthesization generation | 172ms | 152ms | 53s | 56s | 168ms |
| Arithmetic Operator Choice | 6ms | 6ms | 7ms | 11ms | 4ms |
| Scheduling | 29s | 4m21s | 32ms | 132ms | 7s |
| Certification (Gappa) | 6s | 4s | 1m38s | 1m07s | 11s |
| Total time \((\approx)\) | 35s | 4m25s | 2m31s | 2m03s | 18s |

### Timing of each step of the generation flow

- **Impact of the target latency on the first step of the generation**
- **What may dominate the cost**
  - → scheduling algorithm
  - → certification using Gappa
Outline of the talk

1. Design and implementation of floating-point operators
2. Low latency parenthesization computation
3. Selection of effective evaluation parenthesizations
4. Numerical results
5. Conclusions and perspectives
Conclusions

- Design and implementation of floating-point operators
  - uniform approach for correctly-rounded roots and their reciprocals
  - extension to correctly-rounded division
Conclusions

- Design and implementation of floating-point operators
  - uniform approach for correctly-rounded roots and their reciprocals
  - extension to correctly-rounded division
  - polynomial evaluation-based method, very high ILP exposure

⇒ new, much faster version of FLIP
Conclusions

- Design and implementation of floating-point operators
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  - extension to correctly-rounded division
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⇒ new, much faster version of FLIP

- Code generation for efficient and certified polynomial evaluation
  - methodologies and tools for automating polynomial evaluation implementation
  - heuristics and techniques for generating quickly efficient and certified C codes

⇒ CGPE: allows to write and certify automatically $\approx 50\%$ of the codes of FLIP
Perspectives

- Faithful implementation of floating-point operators
  - other floating-point operators:
    - $\log_2(1 + x)$ over $[0.5, 1)$, $1/\sqrt{1 + x^2}$ over $[0, 0.5)$, ...
  - roots and their reciprocals: rounding condition decision not automated yet
Perspectives

- Faithful implementation of floating-point operators
  → other floating-point operators:
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- Extension to other binary floating-point formats
  → square root in $binary64$: 171 cycles on ST231, 396 cycles with STlib
Perspectives

- Faithful implementation of floating-point operators
  - other floating-point operators:
    - \( \log_2(1 + x) \) over \([0.5, 1)\), \(1/\sqrt{1 + x^2}\) over \([0, 0.5)\), ...
  - roots and their reciprocals: rounding condition decision not automated yet

- Extension to other binary floating-point formats
  - square root in \(\text{binary64}\): 171 cycles on ST231, 396 cycles with STlib

- Extension to other architectures, typically FPGAs
  - polynomial evaluation-based approach: already seems to be a good alternative to multiplicative methods on FPGAs
  - the other techniques introduced of this thesis: should be investigated further
Implementation of binary floating-point arithmetic on embedded integer processors
Polynomial evaluation-based algorithms and certified code generation

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Advisors: Claude-Pierre Jeannerod and Gilles Villard

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