Topology of toroidal chaos. A first step.

ISCDE, Batz-sur-mer, October 17-21, 2011

Martin Rosalie & Christophe Letellier

CORIA – Université & INSA de Rouen

Introduction

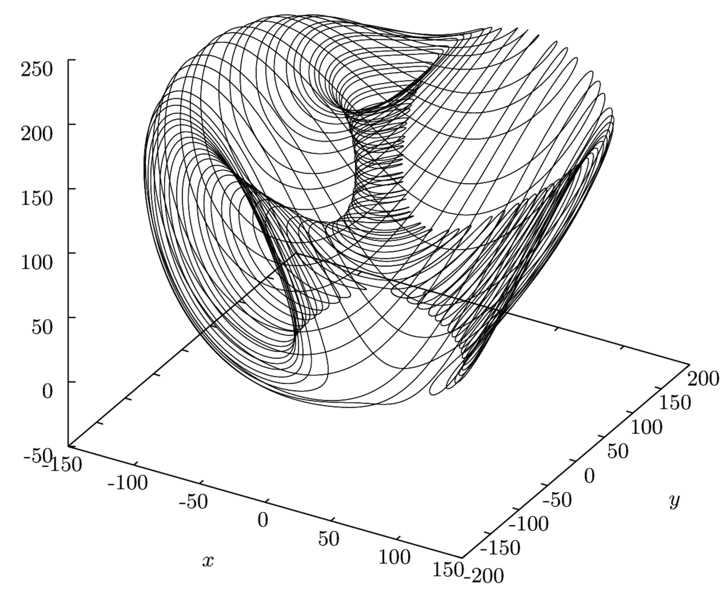
There is little research about the topology of toroidal chaos mainly because a partition, which simplifies the description of unstable periodic orbits, is rather hard to define in this case. Moreover, toroidal surface have no boundary, a property required for constructing branched manifold [1]. Although topological analysis is highly efficient in three-dimensional space, only a few examples of three-dimensional toroidal chaotic systems are known [2,3]. An initial study of the Li system took into account its symmetry properties [4]: an intersection between the symmetry axis and the image attractor – a representation of the attractor where the symmetry is modded out – leads to a non-trivial genus-three torus [5]. We choose a new method to define the Poincaré section for genus-one toroidal attractor, by extending the bounding tori introduced by Tsankov and Gilmore [6]. We started to choose parameter values corresponding to a banded-chaotic attractor (not too far after a period-doubling cascade). Using a coordinate transformation, we unfolded periodic orbits and computed linking numbers to establish the template of the attractor.

Li system

The Li system is a three dimensional system which produces toroidal chaos :

$$\begin{cases} \dot{x} = a(y-x) + dxz \\ \dot{y} = kx + fy - xz \\ \dot{z} = cz + xy - ex^2 \end{cases}$$

For most of the parameter values, trajectories are structured around a non trivial genus-three torus [4] (here exemplified for $a = 41$, $\mathrm{k} = 55$, $\mathrm{f} = 20$, $\mathrm{e} = 0{,}65$, $d = 0{,}16$ and $\mathrm{c} = 11/6$).



Original attractor structured around a non trivial genus-three torus.

The rotation symmetry around the *z*-axis is modded out using a coordinate transformation [5] :

$$\left| \begin{array}{l} u = \mathrm{Re}(\tilde{x} + i\tilde{y})^2 = \tilde{x} + \tilde{y} \\ v = \mathrm{Im}(\tilde{x} + i\tilde{y})^2 = 2\tilde{x}\tilde{y} \\ w = z \end{array} \right.$$

The trajectories of the image system are structured around a genus-one torus.

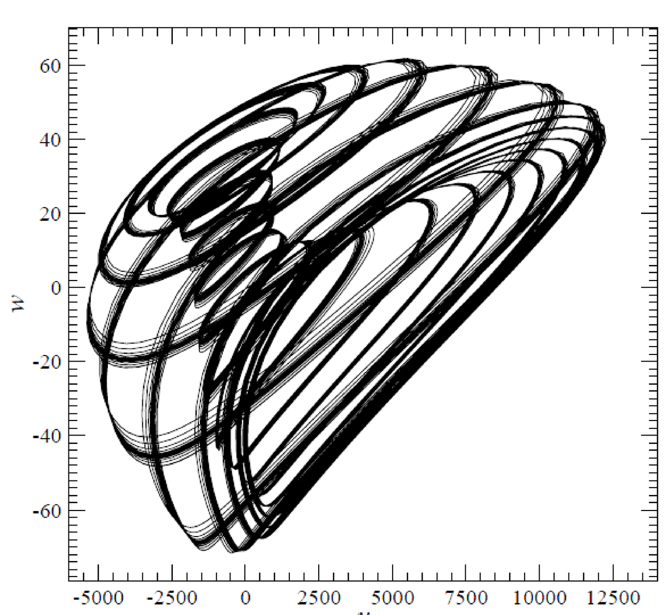


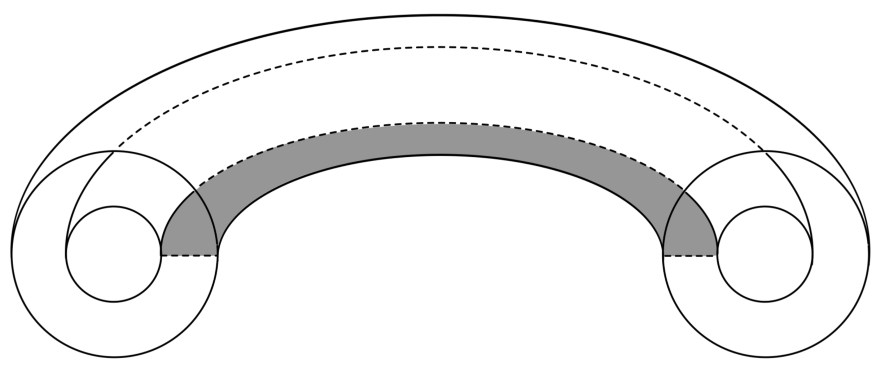
Image attractor in the $u-w$ projection

Poincaré section in the Li image system

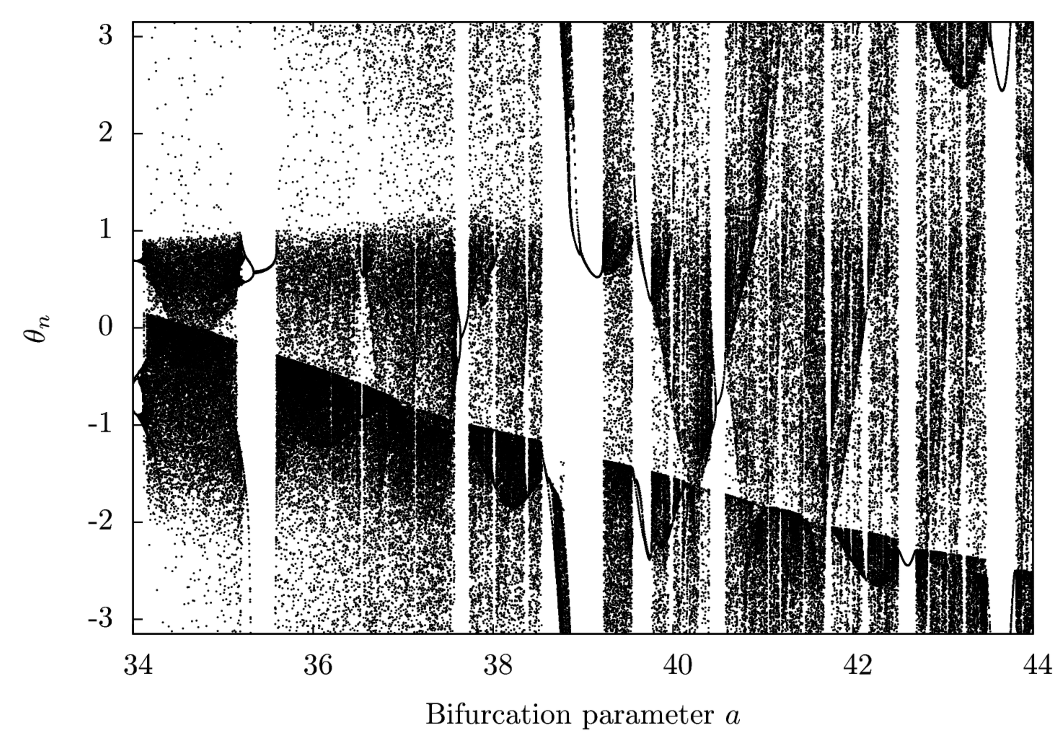
Having extended the theory of bounding tori developed by Tsankov and Gilmore [6] to system bounded by two tori, a Poincaré section was defined between the two bounding tori (see below), that is, by

$$P \equiv \left\{ (u_n, v_n) \in \mathbb{R}^2 \mid w_n = 0, \sqrt{{u_n}^2 + {v_n}^2} < 30 \right\}.$$

The chosen Poincaré section, in grey, between the two bounding tori.

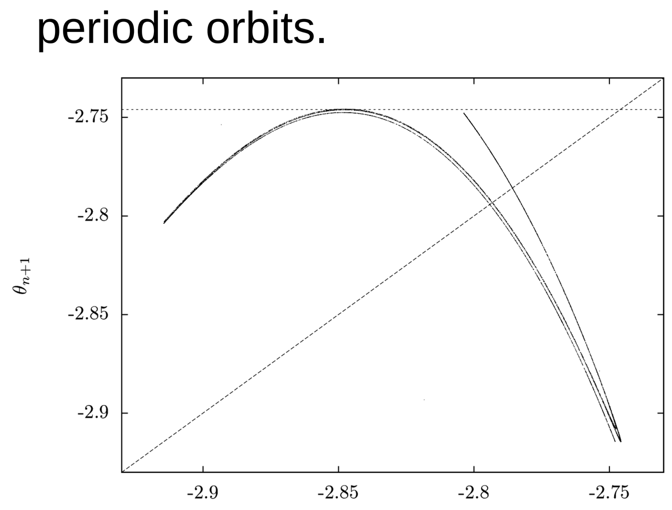


From this section, a bifurcation diagram was built with the angular variable $\theta_n = 2\,\mathrm{atan}(u_n/(v_n + \sqrt{{u_n}^2 + {v_n}^2}))$.

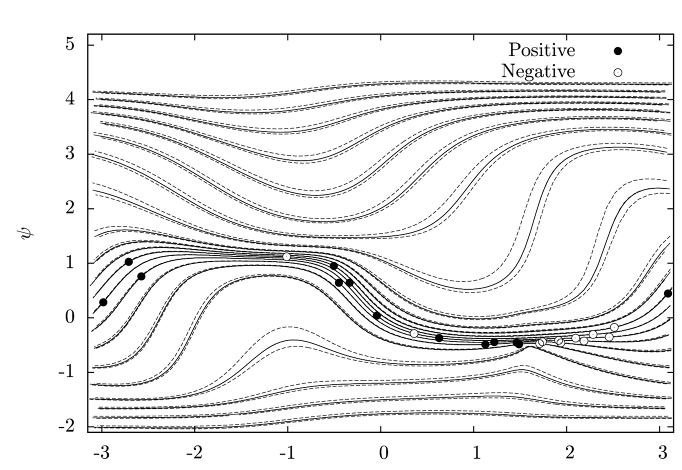


Topology of the image system

A first-return map was built on θ_n. Then periodic orbits were extracted and unfolded using a coordinate transformation (*cf. an example on the Rössler system in the right panel*). Using a third coordinate, linking numbers were computed for few couples of periodic orbits.



First-return map for $a = 35.14$

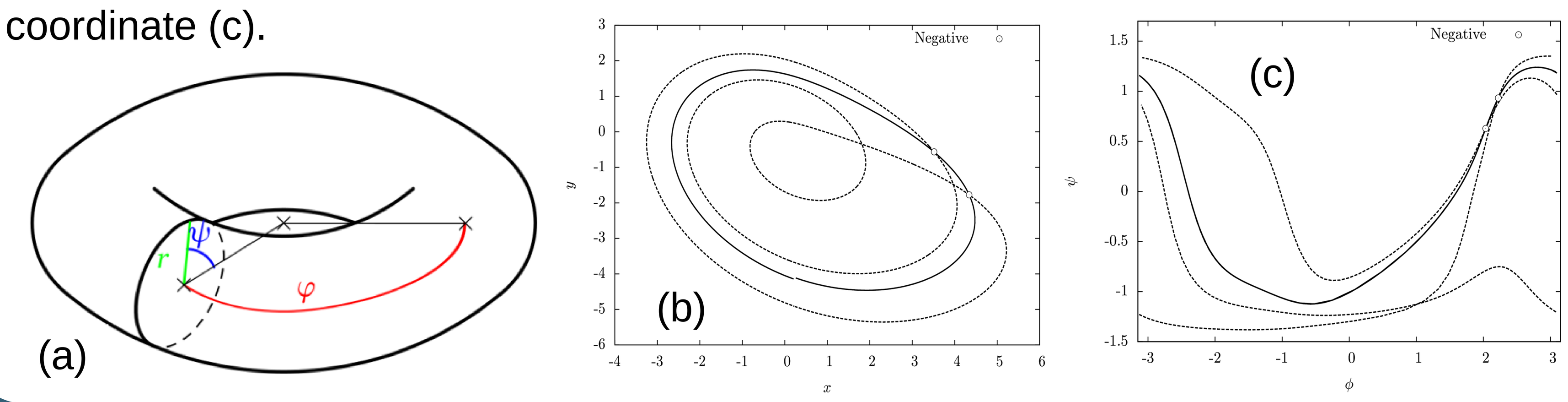


Unfolded unstable periodic orbit

Computed linking numbers can be synthetized in a template (a branched manifold) of the attractor.

Analysis on the Rössler system

A coordinate transformation (a) enables to unfold unstable periodic orbits (b), here extracted from the Rössler attractor [6]. The coordinate transformation maps a torus into a plane. Linking numbers are then computed in this (regular) plane and a third coordinate (c).



The template was encoded by a linking matrix (standard convention as introduced by Tufillaro et al.). We obtained :

$\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$ for the Rössler system and $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ +a global rotation for the Li system

These matrices show that the two systems are topologically equivalent modulo the rotation sign and a global torsion.

Conclusion

We described the topology of a toroidal attractor solution to a three-dimensional system. We modded out the symmetry with a coordinate transformation and obtained the so-called image attractor structured around a genus-one torus. Using the bifurcation diagram, we extracted a first-return map of a banded-chaotic attractor, just after a period-doubling cascade. Periodic orbits were extracted and few linking numbers were computed to identify the template. It was found to be equivalent to the template of the Rössler attractor, modulo the rotation sign and a global torsion.

References

[1] Mindlin, GB & Solari, H.G., Tori and Klein bottles in four-dimensional chaotic flows, *Physica D*, 102 (3-4), 177–186, 1996

[2] Bo Deng, Constructing homoclinic orbits and chaotic attractors, *International Journal of Bifurcation & Chaos*, **4** (4), 823-841, 1994.

[3] D. Li, A three-scroll chaotic attractor, *Physics Letters A,* **372** (4), 387-393, 2008.

[4] C. Letellier & R. Glimore, Poincaré sections for a new three-dimensionnal toroidal attractor, *Journal of Physics A*, **42**, 015101 (2009).

[5] C. Letellier & R. Gilmore. Covering dynamical systems : Two-fold covers, *Physical Review E*, **63**, 16206, 2001.

[6] T.D. Tsankov & R. Gilmore. Topological aspects of the structure of chaotic attractors in R^3. *Physical Review E*, **69** (5), 056206, 2004.

[7] O. E. Rössler, An equation for continuous chaos, *Physics Letters A*, **57** (5), 397, 1976.

UMR 6614 CORIA COMPLEXE DE RECHERCHE INTERPROFESSIONNEL EN AEROTHERMOCHIMIE
CNRS
UNIVERSITÉ DE ROUEN
ATOMOSYD